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Polynomials with the half-plane property and the support theorems

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Abstract

A polynomial $P(\mathbf{x})$ in n complex variables is said to have the half-plane property if $P(\mathbf{x}) \neq 0$ whenever all the variables have positive real parts. The generating polynomial for the set of all spanning trees of a graph G is one example. Motivated by the fact that the edge set of each spanning tree of G is a basis of the graphic matroid induced by G, it is shown by Choe et al. (Adv. Appl. Math. 32 (2004) 88–187) that the support of any homogeneous multiaffine polynomial with the half-plane property constitutes the set of all bases of a matroid. In this paper we show, when all the terms of a polynomial with the half-plane property have degrees of same parity, the support constitutes a jump system which is a generalization of matroids. Open problems and a few directions for further research will also be discussed.

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1. Introduction

Let $P(\mathbf{x})$ be a polynomial in complex variables with complex coefficients. We say that $P(\mathbf{x})$ has the half-plane property if $P(\mathbf{x})$ never vanishes whenever all the variables lie in the open right half-plane of the complex plane. In the joint paper with Oxley et al. [3], we

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showed the following theorem which says that the half-plane property of a polynomial gives rise to the structure of a matroid:

Theorem 1 (*Matroidal Support Theorem, Choe et al. [3]*). *Let E be a finite set and let $P(\mathbf{x}) := \sum_{A\subseteq E} a_A \mathbf{x}^A$ be a homogeneous multilinear polynomial in n complex variables where $a_A \in \mathbb{C}$. If $P(\mathbf{x})$ has the half-plane property then Supp $P(\mathbf{x}) := \{A \subseteq E : a_A \neq 0\}$ is the set of all bases of some matroid.*

As a generalization of matroids, there is a multi-set system called a jump system which was first introduced by Bouchet and Cunningham [1]. In this paper, we generalize Theorem 1 to get the following result:

Theorem 2. *Let $P(\mathbf{x}) := \sum_{\mathbf{m}:E\to\mathbb{N}} a_{\mathbf{m}} \mathbf{x}^{\mathbf{m}}$ be a polynomial where the degree of each term has the same parity. If $P(\mathbf{x})$ has the half-plane property, then the set Supp $P(\mathbf{x}) := \{\mathbf{m} : a_{\mathbf{m}} \neq 0\}$ in $\mathbb{Z}^E$ corresponds to a jump system.*

Theorem 1 then follows as a corollary of Theorem 2. In Section 2, we define basic terms such as jump system and same-phase property and introduce some results from the matroid theory and joint paper [3] which will be used to prove the main theorem. Section 3 contains the main theorem and its proof. We first sketch the proof before going into details. Open problems will be discussed in the last section.

We shall assume that readers are familiar with the basic background in matroid theory [12,15,18]. We will often refer to Oxley [11] for definitions and properties of matroids.

2. Preliminaries

Let E be a totally ordered finite set with n elements. We consider a polynomial $P(\mathbf{x})$ in n complex variables $\{x_e : e \in E\}$ with arbitrary complex coefficients. Then we can express $P(\mathbf{x})$ as follows:

$$P(\mathbf{x}) := \sum_{\mathbf{m}:E\to\mathbb{N}} a_{\mathbf{m}} \mathbf{x}^{\mathbf{m}},$$

where the multi-index $\mathbf{m}$ is a function from E to the set of nonnegative integers $\mathbb{N}$, and $\mathbf{x}^{\mathbf{m}} := \prod_{e\in E} x_e^{\mathbf{m}(e)}$.

We call the finite set E the *ground set of* $P(\mathbf{x})$ and denote it by $E(P)$. The set of multi-indices $\{\mathbf{m} : a_{\mathbf{m}} \neq 0\}$ is called the *support of* $P(\mathbf{x})$ and denoted by *Supp* $P(\mathbf{x})$. Here we assume *Supp* $P(\mathbf{x})$ is always finite.

Definition 3. Let $P(\mathbf{x})$ be a polynomial in complex variables. $P(\mathbf{x})$ is said to have the *half-plane property* (HPP) if $P(\mathbf{x}) \neq 0$ whenever the real part of each variable has a strictly positive value.

From now on, we will often use the abbreviation, HPP, for the half-plane property. We denote the real part of a complex number c by Re c and for $\mathbf{c} \in \mathbb{C}^n$, we denote by Re $\mathbf{c} > 0$ when each entry of $\mathbf{c}$ has a positive real part.

Definition 4. Let a and b be two complex numbers. We say that a and b have the *same phase* if there exists $\theta \in \mathbb{C}$ such that $a = r_1\theta$ and $b = r_2\theta$ for some positive real numbers r_1, r_2.

Let $P(\mathbf{x}) := \sum_{\mathbf{m}:E\to\mathbb{N}} a_{\mathbf{m}}\mathbf{x}^{\mathbf{m}}$, where $a_{\mathbf{m}} \in \mathbb{C}$. We say that the polynomial has the *same phase property* if all the nonzero coefficients have the same phase.

For $\mathbf{m} \in \mathit{Supp}\, P(\mathbf{x})$, the *degree of* $\mathbf{m}$ is the sum $\sum_{e\in E} \mathbf{m}(e)$. When the degree of each term $\mathbf{x}^{\mathbf{m}}$ is either even for all $\mathbf{m} \in \mathit{Supp}\, P(\mathbf{x})$ or odd for all $\mathbf{m} \in \mathit{Supp}\, P(\mathbf{x})$, we say that $P(\mathbf{x})$ has *definite parity*.

The half-plane property of a polynomial gives quite a restriction on the coefficients as you can see from the following theorem:

Theorem 5 (*Same phase property, Choe et al.* [3]). *Let $P(\mathbf{x}) := \sum_{\mathbf{m}} a_{\mathbf{m}}\mathbf{x}^{\mathbf{m}}$ be a polynomial in complex variables with definite parity, where $a_{\mathbf{m}} \in \mathbb{C}$. If $P(\mathbf{x})$ has the half-plane property then all the coefficients have the same phase.*

We can also write the multi-index $\mathbf{m}$ as an n-tuple $(\mathbf{m}(e_1), \ldots, \mathbf{m}(e_n))$ in $\mathbb{Z}^n$. Hence, for $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^n$, addition and subtraction are defined in the following way:

$$(\mathbf{m}+\mathbf{n})(e) = \mathbf{m}(e) + \mathbf{n}(e),$$
$$(\mathbf{m}-\mathbf{n})(e) = \mathbf{m}(e) - \mathbf{n}(e) \quad \text{for any } e \in E.$$

We define $\mathbf{m} \wedge \mathbf{n}$ as

$$(\mathbf{m} \wedge \mathbf{n})(e) = \min\{\mathbf{m}(e), \mathbf{n}(e)\} \quad \text{for any } e \in E$$

and the norm of $\mathbf{m}$ is defined by

$$\|\mathbf{m}\| = \sum_{e\in E} |\mathbf{m}(e)|.$$

Hence, the *degree of* $\mathbf{x}^{\mathbf{m}}$ is $\|\mathbf{m}\|$. The degree of the polynomial $P(\mathbf{x})$ is defined to be $\max\{\|\mathbf{m}\| : \mathbf{m} \in \mathit{Supp}\, P(\mathbf{x})\}$ and we denote it by $deg\, P$. We also define the degree $deg_e\, P$ of a particular variable x_e in $P(\mathbf{x})$ to be the maximum of $\mathbf{m}(e)$ for all $\mathbf{m} \in \mathit{Supp}\, P(\mathbf{x})$. When $deg_e\, P \leqslant 1$ we say that the polynomial is *affine* in e. If P is affine in every $e \in E$, P is said to be *multiaffine* (or multilinear). Hence, when P is multiaffine, we can express $P(\mathbf{x})$ as $\sum_{S\subseteq E} a_S\mathbf{x}^S$ and we have $\mathit{Supp}\, P(\mathbf{x}) = \{S \subseteq E : a_S \neq 0\}$. If the degree of $\mathbf{x}^{\mathbf{m}}$ is either even for each $\mathbf{m} \in \mathit{Supp}\, P(\mathbf{x})$ or odd for every $\mathbf{m} \in \mathit{Supp}\, P(\mathbf{x})$, we say that P has *definite parity*.

When $\mathbf{m}$ and $\mathbf{n}$ are in $\mathbb{Z}^n$, the distance between $\mathbf{m}$ and $\mathbf{n}$ is defined by

$$d(\mathbf{m}, \mathbf{n}) = \|\mathbf{m} - \mathbf{n}\|.$$

We say that $\mathbf{u} \in \mathbb{Z}^n$ is a *step from* $\mathbf{m}$ *to* $\mathbf{n}$ if $\|\mathbf{u}\| = 1$ and $d(\mathbf{m}+\mathbf{u}, \mathbf{n}) = d(\mathbf{m}, \mathbf{n}) - 1$. The set of all steps from $\mathbf{m}$ to $\mathbf{n}$ is denoted by $St(\mathbf{m}, \mathbf{n})$. Now we define a jump system.

Definition 6 (*Jump system, Bouchet and Cunningham* [1]). *Let $\mathcal{J} \subset \mathbb{Z}^n$. $\mathcal{J}$ is called a jump system* if $\mathcal{J}$ satisfies the following axiom.

2-*step axiom*: If $\mathbf{m}_1,\ \mathbf{m}_2 \in \mathcal{J}, \mathbf{u} \in St(\mathbf{m}_1, \mathbf{m}_2)$, and $\mathbf{m}_1 + \mathbf{u} \notin \mathcal{J}$, then there exists $\mathbf{v} \in St(\mathbf{m}_1 + \mathbf{u}, \mathbf{m}_2)$ with $\mathbf{m}_1 + \mathbf{u} + \mathbf{v} \in \mathcal{J}$.

Note if a jump system $\mathcal{J}$ is a subset of $\{0, 1\}^n$, then each vector $\mathbf{m}$ in $\mathcal{J}$ can be considered as the characteristic vector of a subset $U(\mathbf{m})$ of E and we call $\mathcal{J}$ a *delta matroid* [9]. If in addition every element of $\mathcal{J}$ has a constant norm, then the 2-step axiom is equivalent to the basis-exchange axiom. Thus in such a case, the jump system corresponds to the collection of all bases of a matroid.

We found it easier to deal with multiaffine polynomials than nonmultiaffine polynomials. Thus in proving Theorem 2, we use a tool called **r**-*fold polarization of a polynomial* which makes a nonmultiaffine polynomial into a partially symmetric multiaffine polynomial [10].

We denote the kth *elementary symmetric function* on n variables $\{x_1, \ldots, x_n\}$ by

$$E_k(x_1, x_2, \ldots, x_n) = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n} x_{i_1} x_{i_2} \cdots x_{i_k},$$

where E_0 is defined to be 1. Note that any linear combination of elementary symmetric functions is symmetric and multiaffine.

Given a univariate polynomial $P(x) := \sum_{k=0}^{r} a_k x^k$ of degree r, we define a multiaffine polynomial $\mathcal{P}_x^r(P)$ in r variables as follows:

$$\mathcal{P}_x^r(P) := \sum_{k=0}^{r} a_k \binom{r}{k}^{-1} E_k(x_1, x_2, \ldots, x_r).$$

We call $\mathcal{P}_x^r(P)$ the *r-fold polarization of x in P*.

Definition 7 (*Polarization*). Let $P(x_1, x_2, \ldots, x_n)$ be a polynomial in n variables. Let $r_i := deg_i\ P$. Then the *polarization* of P is the multiaffine polynomial in variables $\bigcup_{i=1}^{n} \{(x_i)_j : j = 1, 2, \ldots, r_i\}$ which is defined by

$$\mathcal{P}(P) := \left(\prod_{i=1}^{n} \mathcal{P}_{x_i}^{r_i} \right) P(x_1, x_2, \ldots, x_n).$$

Example 8. Let $P(x_1, x_2) = x_1^3 x_2 + 5x_1^2 + x_2^2$. Then we have

$$\begin{aligned}\mathcal{P}(P) = {}& (x_{11}x_{12}x_{13})(1/2)(x_{21} + x_{22}) + (5/3)(x_{11}x_{12} \\ & + x_{11}x_{13} + x_{12}x_{13}) + x_{21}x_{22}.\end{aligned}$$

Hence, for any given polynomial $P(\mathbf{x})$ in n variables, $\{x_1, x_2, \ldots, x_n\}$, we can construct the unique multiaffine polynomial $\mathcal{P}(P)$ which is symmetric under all permutations of $\{(x_i)_j : j = 1, 2, \ldots, deg_i\ P\}$ separately for each $i = 1, 2, \ldots, n$. Using this fact, we then apply Grace–Walsh–Szegö Coincidence Theorem [3, Theorem 2.12, 17] to get the following result:

Proposition 9 (*Choe et al.* [3] *and Choe* [5]). *Let $P(\mathbf{x})$ be a polynomial in n complex variables. Then $P(\mathbf{x})$ has the half-plane property if and only if its polarization $\mathcal{P}(P)$ has the half-plane property*.

We now know that polarization on a polynomial preserves the HPP.

There are a few more properties of the polynomials with the HPP needed to prove the main result in the following section. We refer to [3] for proofs.

First, we define *deletion* and *contraction* of a polynomial in a similar way as we defined for matroids.

Definition 10 (*Deletion and contraction*). Let $P(\mathbf{x})$ be a polynomial in complex variables on the ground set E. For any $e \in E$,

(a) $P^{\backslash e} := P(\mathbf{x})|_{x_e=0}$ is the polynomial on the ground set $E\backslash e$ obtained from P by replacing x_e by zero. $P^{\backslash e}$ is called the *deletion* of e from P.
(b) The *contraction* of e from P is defined as the polynomial $\partial P/\partial x_e$ and denoted by $P^{/e}$.

Both deletion and contraction are associative and commutative operation so that we can further define $P^{\backslash S}(\mathbf{x})$ and $P^{/S}(\mathbf{x})$ for any subset $S \subseteq E$.

Definition 11 (*Dual polynomial*). For a polynomial $P(\mathbf{x}) = \sum_{S\subseteq E} a_S\mathbf{x}^S$ which is multiaffine, the *dual polynomial* is defined as follows:

$$P^*(\mathbf{x}) = \sum_{S\subseteq E} a_S\mathbf{x}^{E\backslash S}. \tag{1}$$

The HPP is preserved under the operations we defined above [3].

Proposition 12 (*Choe et al.* [3, Proposition 3.1]). *Let P be a polynomial with the half-plane property. Then, for every $e \in E(P)$, $P^{\backslash e}$ and $P^{/e}$ both have the half-plane property*.

Since E is a finite set, the following property holds.

Corollary 13. *Let P be a polynomial with the half-plane property. Then for any subset $S \subseteq E(P)$, $P^{\backslash S}$ and $P^{/S}$ both have the half-plane property*.

Proposition 14 (*Choe et al.* [3, Proposition 4.2]). *If a multiaffine polynomial P has the half-plane-property, then so does the dual polynomial P^**.

3. The half-plane property and the combinatorial structure behind it

In this section, we prove a few lemmas and a theorem to eventually show the jump system support theorem.

Theorem 15 (*Jump-system support theorem*). *Let $P(\mathbf{x})$ be a polynomial in n complex variables $\{x_1, x_2, \dots, x_n\}$ with complex coefficients which has definite parity. If $P(\mathbf{x})$ has the half-plane property, then Supp $P(\mathbf{x})(\subseteq \mathbb{Z}^n)$ is a jump system.*

From Section 2, we know that the HPP is preserved by polarization. To prove Theorem 2, we first polarize the given polynomial P and get the same result for $\mathcal{P}(P)$. Hence, what we mainly prove here is the following:

Theorem 16. *Let $P(\mathbf{x}) = \sum_{U \subseteq E} a_U \mathbf{x}^U$ be a multiaffine polynomial in n complex variables with definite parity, where $a_U \in \mathbb{C}$. If $P(\mathbf{x})$ has the half-plane property, then Supp $P(\mathbf{x})(\subseteq \mathbb{Z}^E)$ is a jump system.*

We prove Theorem 16 by induction on the degree of the polynomial $P(\mathbf{x})$. In the induction step, we show the 2-step axiom for every pair (A, B) of sets A, B in *Supp* $P(\mathbf{x})$. We use deletion and contraction on $P(\mathbf{x})$ to apply the induction hypotheses.

Before we begin the proof, we prove several Lemmas that appear often in later arguments so that we do not need to repeat the same proofs over and over again.

Let $\mathbf{0}$ denote the zero vector and $\mathbf{1}$ the vector all of whose entries are 1.

Lemma 17. *Let $P(\mathbf{x}) = \sum_{\alpha: E \to \mathbb{N}} a_\alpha \mathbf{x}^\alpha$ be a polynomial in n complex variables with definite parity, where $a_\alpha \in \mathbb{C}$. Suppose that $P(\mathbf{x})$ has the half-plane property and let $\alpha, \beta \in$ Supp $P(\mathbf{x})$ and $\alpha \neq \mathbf{0} \neq \beta$. If $\alpha \wedge \beta = \mathbf{0}$ and $\alpha + \beta \geqslant \mathbf{1}$, then there exists $\gamma \in$ Supp $P(\mathbf{x})$ s.t. $\alpha \wedge \gamma \neq \mathbf{0} \neq \gamma \wedge \beta$.*

Proof. By Theorem 5, we may assume that $a_\alpha > 0$ for any $\alpha \in$ *Supp* $P(\mathbf{x})$.

Define $X, Y \subseteq E$ as follows:

$$X := \{e \in E : \alpha(e) \neq 0\} \quad \text{and} \quad Y := \{e \in E : \beta(e) \neq 0\}.$$

Clearly, $\{X, Y\}$ is a partition of E. Suppose that there exists no such γ. Then for any $\delta \in$ *Supp* $P(\mathbf{x})$, we have either $\delta \wedge \alpha = \mathbf{0}$ or $\delta \wedge \beta = \mathbf{0}$. Hence, we can break the terms of $P(\mathbf{x})$ into two parts, say $Q(\mathbf{x})$ and $R(\mathbf{x})$, so that we have $P(\mathbf{x}) = Q(\mathbf{x}) + R(\mathbf{x})$ and

$$\bigcup_{\delta \in \mathit{Supp}\, Q(\mathbf{x})} \{e \in E : \delta(e) \neq 0\} = X \quad \text{and} \quad \bigcup_{\delta \in \mathit{Supp}\, R(\mathbf{x})} \{e \in E : \delta(e) \neq 0\} = Y.$$

We define the univariate polynomials $q(u)$ and $r(v)$ from $Q(\mathbf{x})$ and $R(\mathbf{x})$ as follows:

$$q(u) := Q(\mathbf{x})|_{x_e = u,\, e \in X} \quad \text{and} \quad r(v) := R(\mathbf{x})|_{x_e = v,\, e \in Y}.$$

There are two cases to consider:

(i) If $P(\mathbf{x})$ is an *even* polynomial, then $Q(\mathbf{x})$, $R(\mathbf{x})$ are even polynomials and so are $q(u)$ and $r(v)$. By the fundamental theorem of algebra, zeros of the polynomials $q(u) - i$ and $r(v) + i$ exist. Let $u_0,\ v_0$ be roots of the equations $q(u) - i = 0$ and $r(v) + i = 0$, respectively. Note that neither Re u_0 nor Re v_0 is zero, for $q(u), r(v)$ are even polynomials. Moreover, we can always find the roots in with positive real parts since if Re $u_0 < 0$ then

$\mathrm{Re}\,(-u_0) > 0$ and $q(-u_0) = q(u_0) = 0$. Likewise, we can always find v_0 such that $\mathrm{Re}\, v_0 > 0$. Hence, we may assume that $\mathrm{Re}\, u_0, \mathrm{Re}\, v_0 > 0$.
Define $\mathbf{x}^*$ by

$$x_e^* := \begin{cases} u_0 & \text{if } e \in X, \\ v_0 & \text{if } e \in Y. \end{cases} \tag{2}$$

Then, clearly $\mathrm{Re}\, x_e{}^* > 0$, for any e and we have $P(\mathbf{x}^*) = q(u_0) + r(v_0) = i - i = 0$. This contradicts the hypothesis that $P(\mathbf{x})$ has the HPP.

(ii) If $P(\mathbf{x})$ is an *odd* polynomial, then so are $Q(\mathbf{x})$, $R(\mathbf{x})$, $q(u)$, and $r(v)$. Let u_0, v_0 be solutions of the equations $q(u) = i - 1$ and $r(v) = -i + 1$, respectively. Since $q(u)$ and $r(v)$ are odd polynomials, if $\mathrm{Re}\, u_0 = 0$ or $\mathrm{Re}\, v_0 = 0$, then $\mathrm{Re}\, q(u_0) = 0$ or $\mathrm{Re}\, r(v_0) = 0$, respectively. Hence $\mathrm{Re}\, u_0$ and $\mathrm{Re}\, v_0$ cannot be zero. Moreover, since $d_1 := deg\, q(u)$ and $d_2 := \deg r(v)$ are both odd, the coefficients of u^{d_1-1} and v^{d_2-1} in $q(u)$ and $r(v)$ are both zero, respectively. Those coefficients represent the sums of all the solutions of $q(u) - (1 - i) = 0$ and $r(v) - (-i + 1) = 0$. Hence, there must exist u_0 and v_0 satisfying $\mathrm{Re}\, u_0,\ \mathrm{Re}\, v_0 > 0$. Then with $\mathbf{x}^*$ defined as in (2), we have $P(\mathbf{x}^*) = 0$ and it again contradicts the HPP of $P(\mathbf{x})$.

Therefore, by (i) and (ii), there exists $\gamma \in$ *Supp* $P(\mathbf{x})$ s.t. $X \cap Z \neq \emptyset \neq Z \cap Y$ where $Z := \{e \in E : \gamma(e) \neq 0\}$. $\square$

Corollary 18. *Let $P(\mathbf{x}) = \sum_{U \subseteq E} a_U \mathbf{x}^U$ be a multilinear polynomial in n complex variables with definite parity, where $a_U \in \mathbb{C}$. Suppose that $P(\mathbf{x})$ has the half-plane property and two nonempty sets, A and B, in Supp $P(\mathbf{x})$ partition the ground set E. Then there exists $C \in$ Supp $P(\mathbf{x})$ s.t. $A \cap C \neq \emptyset \neq C \cap B$.*

We denote the collection of all such $C \in$ *Supp* $P(\mathbf{x})$ in the above corollary by $\mathcal{C}[A, B]$.

Lemma 19. *Let E be a set with n elements and let $\mathcal{J}$ be a collection of some subsets of E. Suppose that $|A| \equiv |B| \pmod 2$ for any $A, B \in \mathcal{J}$. Then $\mathcal{J}$ is a jump system if and only if any two sets A and B in $\mathcal{J}$ satisfy the following*:

(i) *for any $a \in A \setminus B$, there exists $a' \in (A \setminus a) \setminus B$ such that $A \setminus \{a, a'\} \in \mathcal{J}$ or there exists $b' \in B \setminus A$ such that $(A \setminus a) \cup b' \in \mathcal{J}$.*
(ii) *for any $b \in B \setminus A$, there exists $a'' \in A \setminus B$ such that $(A \cup b) \setminus a'' \in \mathcal{J}$ or there exists $b'' \in (B \setminus b) \setminus A$ such that $A \cup \{b, b''\} \in \mathcal{J}$.*

Proof. We can consider every subset A of E in $\mathcal{J}$ as a characteristic vector $\mathbf{m} \in \mathbb{Z}^E$ whose eth coordinate is 1 if $e \in A$ or 0 otherwise. Then it is clear that $\mathcal{J} \subseteq \{0, 1\}^E$. For $e \in E$, let δ_e denote the vector in $\mathbb{N}^E$ whose eth entry is 1 and the rest are 0. $\mathcal{J}$ is a jump system if and only if $\mathcal{J}$ satisfies the 2-step axiom. Let α and β be vectors in $\{0, 1\}^E$ that correspond to A and B in $\mathcal{J}$, respectively, and $\mathbf{u}$ be a step from α to β. Then, $St(\alpha, \beta) = \{-\delta_a : a \in A \setminus B\} \cup \{\delta_b : b \in B \setminus A\}$. Thus we need to check the following two cases:

(i′) $\mathbf{u} = -\delta_a$ for some $a \in A \setminus B$:

since $\|\alpha - \delta_a\|$ and $\|\alpha\|$ have different parity, $\alpha - \delta_a \notin \mathcal{J}$. Then there must exist $\mathbf{v} \in St(\alpha - \delta_a, \beta)$ such that $\alpha - \delta_a + \mathbf{v} \in \mathcal{J}$. Again, since $\alpha - \delta_a, \beta \in \{0, 1\}^E$, $\mathbf{v}$ is either $-\delta_{a'}$ for some $a' \in (A \setminus a) \setminus B$ or $\delta_{b'}$ for some $b' \in B \setminus A$. Hence, $A \setminus \{a, a'\} \in \mathcal{J}$ or $A \setminus a \cup b' \in \mathcal{J}$.

(ii″) $\mathbf{u} = \delta_b$ for some $b \in B \setminus A$:
since $\|\alpha+\delta_b\|$ and $\|\alpha\|$ have different parity, $\alpha+\delta_b \notin \mathcal{J}$. Then there exists $v \in St(\alpha+\delta_b, \beta)$ such that $\alpha + \delta_b + \mathbf{v} \in \mathcal{J}$, where $\mathbf{v}$ is either $-\delta_{a''}$ for some $a'' \in A \setminus B$ or $\delta_{b''}$ for some $b'' \in (B \setminus b) \setminus A$. Hence, $A \cup b \setminus a'' \in \mathcal{J}$ or $A \cup \{b, b''\} \in \mathcal{J}$.

Clearly, (i′) and (ii′) are equivalent to (i) and (ii), respectively. $\square$

Given a multiaffine polynomial $P(\mathbf{x})$ and $A, B \in$ *Supp* $P(\mathbf{x})$, we say that the pair (A, B) satisfies the 2-step axiom in $P(\mathbf{x})$ if and only if (i) and (ii) in Lemma 19 hold. If (i) or (ii) holds for one particular element $e \in E$ or for a subset $S \subseteq E$, we say (A, B) *satisfies the 2-step axiom for e or for S in* $P(\mathbf{x})$, respectively.

Lemma 20. *Let* $P(\mathbf{x}) = \sum_{S\subseteq E} a_S\mathbf{x}^S$ *be a multilinear polynomial in n complex variables with definite parity and complex coefficients. Suppose that* $P(\mathbf{x})$ *has the half-plane property and Supp* $P^{\setminus D}(\mathbf{x})$ *and Supp* $P^{/D}(\mathbf{x})$ *are jump systems for any proper subset D of E. Suppose that* A, B, C *are in Supp* $P(\mathbf{x})$ *and* $A \cup B = E$, $A \cap B = \emptyset$ *and* $C \in \mathcal{C}[A, B]$. *Then*

(i) *for any* $a \in A \setminus C$, (A, B) *satisfies the* 2*-step axiom in* $P(\mathbf{x})$,
(ii) *for any* $b \in C \cap B$, (A, B) *satisfies the* 2*-step axiom in* $P(\mathbf{x})$.

Proof. Since $A, C \in$ *Supp* $P(\mathbf{x})$, $A\setminus(A\cap C)$ and $C\setminus(A\cap C)$ are contained in *Supp* $P^{/A\cap C}(\mathbf{x})$. By the hypotheses, *Supp* $P^{/A\cap C}(\mathbf{x})$ is a jump system and $P^{/A\cap C}(\mathbf{x})$ has definite parity. Then we can apply Lemma 19 to $(A\setminus C, C\setminus A)$ and $P^{/A\cap C}(\mathbf{x})$. Thus we have the following:

(i′) For any $a \in (A \setminus C) \setminus (C \setminus A) = A \setminus C$, there exists $a' \in (A \setminus C) \setminus a$ such that $A \setminus C \setminus \{a, a'\} \in$ *Supp* $P^{/A\cap C}(\mathbf{x})$ or there exists $b' \in C \setminus A$ such that $A \setminus C \setminus a \cup b' \in$ *Supp* $P^{/A\cap C}(\mathbf{x})$. Therefore, there exists $A \setminus \{a, a'\}$ or $(A \setminus a) \cup b'$ in *Supp* $P(\mathbf{x})$ for some $a' \in (A \setminus C) \setminus a \subset A \setminus a$ and $b' \in C \setminus A \subset B$.

(ii″) For any $b \in C \setminus A$, there must exist $a'' \in A \setminus C$ such that $(A \setminus C) \cup b \setminus a'' \in$ *Supp* $P^{/A\cap C}(\mathbf{x})$ or $b'' \in C \setminus A \setminus b$ such that $A \setminus C \cup \{b, b''\} \in$ *Supp* $P^{/A\cap C}(\mathbf{x})$. Therefore $A \cup b \setminus a'' \in$ *Supp* $P(\mathbf{x})$ or $A \cup \{b, b''\} \in$ *Supp* $P(\mathbf{x})$ for some $a'' \in A \setminus C \subset A$ and $b'' \in C \setminus B \setminus b \subset A \setminus b$.

Therefore, we conclude (i) and (ii) from (i′) and (ii′) respectively. $\square$

Lemma 21. *Let* $\mathcal{P}_r$ *be the set of all multiaffine polynomials of degree at most r with n complex variables* $(x_e : e \in E)$ *which have definite parity and the half-plane property. Suppose that for any polynomial* $P(\mathbf{x}) \in \mathcal{P}_r$ *with* $deg\ P < r$, *Supp* $P(\mathbf{x})$ *is a jump system and that for any* $P(\mathbf{x}) \in \mathcal{P}_r$ *of degree r*, (A, B) *satisfies the* 2*-step axiom when* $A, B \in$ *Supp* $P(\mathbf{x})$ *such that* $|A|\leqslant|B|$, $|A|\leqslant 2$, $A \cap B = \emptyset$ *and* $A \cup B = E$. *Then* (B, A) *also satisfies the* 2*-step axiom unless both* $\emptyset \in$ *Supp* $P(\mathbf{x})$ *and* $E \notin$ *Supp* $P(\mathbf{x})$ *hold.*

Proof. Consider the dual polynomial $P^*(\mathbf{x})$ of $P(\mathbf{x})$ which was defined in (1) in Section 2. By Proposition 14, $P^*(\mathbf{x})$ has the half-plane property. Since *Supp* $P^*(\mathbf{x}) = \{E \setminus S :$

$S \in \mathit{Supp}\, P(\mathbf{x})\}$, $P^*(\mathbf{x})$ also has definite parity. Moreover, since $A, B \in \mathit{Supp}\, P(\mathbf{x})$, $E \setminus A = B$ and $E \setminus B = A$ are in $\mathit{Supp}\, P^*(\mathbf{x})$. Also we have $deg\, P^*(\mathbf{x}) \leqslant deg\, P(\mathbf{x})$, when $\emptyset \notin \mathit{Supp}\, P(\mathbf{x})$ or $E \in \mathit{Supp}\, P(\mathbf{x})$. Thus by the hypothesis, (A, B) satisfies the 2-step axiom in $P^*(\mathbf{x})$. That is,

(i) for any $a \in A$, there exists $a' \in A \setminus a$ such that $A \setminus \{a, a'\} \in \mathit{Supp}\, P^*(\mathbf{x})$ or there exists $b' \in B$ such that $A \setminus a \cup b' \in \mathit{Supp}\; P^*(\mathbf{x})$,

(ii) for any $b \in B$, there exists $a'' \in A$ such that $A \cup b \setminus a'' \in \mathit{Supp}\, P^*(\mathbf{x})$ or there exists $b'' \in B \setminus b$ such that $A \cup \{b, b''\} \in \mathit{Supp}\, P^*(\mathbf{x})$.

In (i), $A \setminus \{a, a'\} \in \mathit{Supp}\, P^*(\mathbf{x})$ if and only if $E \setminus (A \setminus \{a, a'\}) = B \cup \{a, a'\} \in \mathit{Supp}\, P(\mathbf{x})$ and $A \setminus a \cup b' \in \mathit{Supp}\, P^*(\mathbf{x})$ if and only if $E \setminus (A \setminus a \cup b') = B \cup a \setminus b' \in \mathit{Supp}\, P(\mathbf{x})$. We can easily get analogous result for (ii). Therefore (i) and (ii) can be rewritten, respectively, as follows:

(ii′) for any $a \in A$, there exists $a' \in A \setminus a$ such that $B \cup \{a, a'\} \in \mathit{Supp}\, P(\mathbf{x})$ or there exists $b' \in B$ such that $B \setminus b' \cup a \in \mathit{Supp}\, P(\mathbf{x})$,

(i′) for any $b \in B$, there exists $a'' \in A$ such that $B \setminus b \cup a'' \in \mathit{Supp}\, P(\mathbf{x})$ or there exists $b'' \in B \setminus b$ such that $B \setminus \{b, b''\} \in \mathit{Supp}\, P(\mathbf{x})$.

Thus (B, A) satisfies the 2-step axiom in $P(\mathbf{x})$. □

Now we are ready to prove the main theorem.

Proof of Theorem 16. By the same phase property (Theorem 5), we may assume that a_U is a positive real number for any $U \in \mathit{Supp}\, P(\mathbf{x})$. Let r be the degree of $P(\mathbf{x})$. We will prove the theorem by induction on r.

First, when $r = 1$, $P(\mathbf{x})$ is homogeneous, i.e., $P(\mathbf{x}) = \sum_{e \in E} a_e x_e$ and thus $\mathit{Supp}\, P(\mathbf{x})$ is trivially a jump system.

Suppose that for the multiaffine polynomials of degree $\leqslant r$ with definite parity and the half-plane property, $\mathit{Supp}\, P(\mathbf{x})$ is a jump system.

As the induction step, we need to show that the theorem holds when the degree is $r + 1$. Assume that the theorem does not hold for a polynomial of degree $r + 1$. Let $P(\mathbf{x})$ be a multiaffine polynomial of degree $r + 1$ with definite parity and the HPP whose support is not a jump system, for which the size of the ground set E is as small as possible. Hence, $P(\mathbf{x})$ is the minimal counter example with respect to r and $|E|$.

For any proper subset $S \subset E$, consider $P^{\setminus S}(\mathbf{x})$ and $P^{/S}(\mathbf{x})$. They both have definite parity and by Corollary 13 they also have the HPP. Since $P^{/S}(\mathbf{x})$ is of degree at most r and the ground set for $P^{\setminus S}(\mathbf{x})$ is of size at most $n - 1$, induction hypothesis applies to both polynomials. Therefore, $\mathit{Supp}\, P^{/S}(\mathbf{x})$ and $\mathit{Supp}\, P^{\setminus S}(\mathbf{x})$ are jump systems for any proper subset $S \subsetneq E$.

Since $\mathit{Supp}\, P(\mathbf{x})$ is not a jump system, there must exist $A, B \in \mathit{Supp}\, P(\mathbf{x})$ for which the 2-step axiom does not hold. We will show that the 2-step axiom holds for any pair (A, B) in $\mathit{Supp}\, P(\mathbf{x})$, which will contradict the assumption that $\mathit{Supp}\, P(\mathbf{x})$ is not a jump system.

We first divide (A, B) into the following three cases:

- $A \cup B \neq E$.
- $A \cup B = E$ and $A \cap B \neq \emptyset$.
- $A \cup B = E$ and $A \cap B = \emptyset$.

Case I: $A \cup B \neq E$.
There exists $e \in E \setminus (A \cup B)$. Consider $P^{\setminus e}(\mathbf{x})$. By induction hypothesis, *Supp* $P^{\setminus e}(\mathbf{x})$ is a jump system. Since $e \notin A$ and $e \notin B$, $A, B \in$ *Supp* $P^{\setminus e}(\mathbf{x})$ and thus (A, B) satisfies the 2-step axiom in $P^{\setminus e}(\mathbf{x})$, i.e.,

(i) For any $a \in A \setminus B$, (A, B) satisfies the 2-step axiom in $P^{\setminus e}(\mathbf{x})$.
(ii) For any $b \in B \setminus A$, (A, B) satisfies the 2-step axiom in $P^{\setminus e}(\mathbf{x})$.

However, *Supp* $P^{\setminus e}(\mathbf{x}) \subseteq$ *Supp* $P(\mathbf{x})$, and thus (i) and (ii) implies that (A, B) satisfies the 2-step axiom in $P(\mathbf{x})$.

Case II: $A \cup B = E$ and $A \cap B \neq \emptyset$.
Let e be an element in $A \cap B$ and consider $P^{/e}(\mathbf{x})$. $A \backslash e$, $B \backslash e \in$ *Supp* $P^{/e}(\mathbf{x})$ and *Supp* $P^{/e}(\mathbf{x})$ is a jump system. Thus $(A \backslash e, B \backslash e)$ satisfies the following 2-step axiom:

(i) For $a \in (A \setminus e) \setminus (B \setminus e)$, $(A \setminus e, B \setminus e)$ satisfies the 2-step axiom in $P^{/e}(\mathbf{x})$.
(ii) For $b \in (B \setminus e) \setminus (A \setminus e)$, $(A \setminus e, B \setminus e)$ satisfies the 2-step axiom in $P^{/e}(\mathbf{x})$.

Since $e \in A \cap B$, $(A \setminus e) \setminus (B \setminus e) = A \setminus B$ and $(B \setminus e) \setminus (A \setminus e) = B \setminus A$. When $e \in S$, S is in *Supp* $P(\mathbf{x})$ if and only if $S \setminus e \in$ *Supp* $P^{/e}(\mathbf{x})$. Hence (i) and (ii) are equivalent to the following:

(i$'$) For $a \in A \setminus B$, (A, B) satisfies the 2-step axiom in $P(\mathbf{x})$.
(ii$''$) For $b \in B \setminus A$, (A, B) satisfies the 2-step axiom in $P(\mathbf{x})$.

Therefore (i$'$) and (ii$'$) imply the 2-step axiom for (A, B) in $P(\mathbf{x})$.

Case III: $A \cup B = E$ and $A \cap B = \emptyset$.
Since $P(\mathbf{x})$ has definite parity, we only need to check the cases of (A, B) such that $|A| \equiv |B| \pmod 2$.

When $|A| = 0, |B| = 2$ or $|A| = |B| = 1$, it is trivial that (A, B) satisfies the 2-step axiom. Hence, there are 5 subcases to consider for (A, B) pair with respect to the size of each set A and B:

(1) $|A| = 0$ and $|B| \geqslant 4$.
  (1.1) $|B| = 4$.
  (1.2) $|B| \geqslant 6$.
(2) $|A| = 1$ and $|B| \geqslant 3$.
  (2.1) $|B| = 3$.
  (2.2) $|B| \geqslant 5$.
(3) $|A| = 2$ and $|B| \geqslant 2$.
  (3.1) $|B| = 2$.
  (3.2) $|B| \geqslant 4$.
(4) $|A| \geqslant 3$ and $|B| \geqslant 3$.
(5) $|A| \geqslant |B|$ and $|B| \leqslant 2$.
  (5.1) $|B| \leqslant 1$.
  (5.2) $|B| = 2$.

Note that this is a part of the induction step and we will keep using the induction hypothesis for the polynomials of smaller degrees or with smaller ground sets to show the 2-step axiom of (A, B), or equivalently, (i) and (ii) of Lemma 19. In some subcases, there are occasions such as (1.1), (2.1) and (3.1) where the induction hypothesis does not help proving the 2-step axiom of the (A, B) pair. For those occasions, we think of all possible cases in which (A, B) does not satisfy the 2-step axiom. We then show that the polynomial $P(\mathbf{x})$ corresponding to each of the possible cases does not have the HPP, by exhibiting a root $\mathbf{x}^*$ of $P(\mathbf{x})$ with $\mathrm{Re}\,\mathbf{x}^* > 0$. This result will contradict our hypothesis that $P(\mathbf{x})$ satisfies the HPP. Thus, we can conclude (A, B) satisfies the 2-step axiom in each of those cases.

Subcase 1: $|A| = 0$ and $|B| \geqslant 4$.
Clearly, $A = \emptyset$ and $B = E$. Let $B := \{b_1, b_2, \ldots, b_{2k}\}$ for some $k \geqslant 2$. The 2-step axiom for (A, B) is equivalent to showing that for each $b \in B$, there exists a 2-subset C in *Supp* $P(\mathbf{x})$ that contain b.

First, we claim that $\{\emptyset, E\}$ is a proper subset of *Supp* $P(\mathbf{x})$. The polynomial $P(\mathbf{x}) := K + Lx_1x_2 \cdots x_{2k}$ for arbitrary positive real numbers, K and L, has a root $x_1 = x_2 = \cdots = x_{2k} = \sqrt[2k]{K/L}\, e^{i\pi/2k}$, where $\mathrm{Re}\ \sqrt[2k]{K/L}\, e^{i\pi/2k} > 0$. This contradicts our assumption that $P(\mathbf{x})$ does not have the HPP. Therefore, there exists another set, say C, in *Supp* $P(\mathbf{x})$.

We consider two cases with respect to $|B|$.

Subcase (1.1): $|B| = 4$.
Let $C := \{b_1, b_2\}$ and suppose that (A, B) does not satisfies the 2-step axiom. Then, there must exist $b \in B$ such that no 2-subset in *Supp* $P(\mathbf{x})$ contains b. Let $b := b_4$. Hence, without loss of generality, we may assume

$$P(\mathbf{x}) = 1 + Lx_1x_2 + Mx_1x_3 + Nx_2x_3 + Kx_1x_2x_3x_4, \tag{3}$$

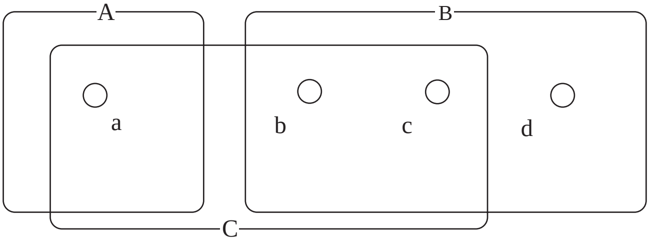
where K, L, M, and N are nonnegative reals and $KL \neq 0$.
Define $\mathbf{x}^* := (x_1^*, x_2^*, x_3^*, x_4^*)$ as follows:

$$\begin{aligned}
x_1^* &= (N+1)/(1+3i),\\
x_2^* &= (M+1)/(6-i),\\
x_3^* &= (2+TLi)/(L), \quad \text{where } T := (2/L) + (N+1)(M+1),\\
x_4^* &= P_0(x_1^*, x_2^*, x_3^*) := -\frac{1}{K}\left(\frac{L}{x_1^*} + \frac{M}{x_2^*} + \frac{N}{x_3^*} + \frac{1}{x_1^*x_2^*x_3^*}\right).
\end{aligned}$$

Then, we have $\mathrm{Re}\,\mathbf{x}^* > 0$ and $P(\mathbf{x}^*) = 0$, which contradicts the HPP of $P(\mathbf{x})$. Therefore, (A, B) satisfies the 2-step axiom.

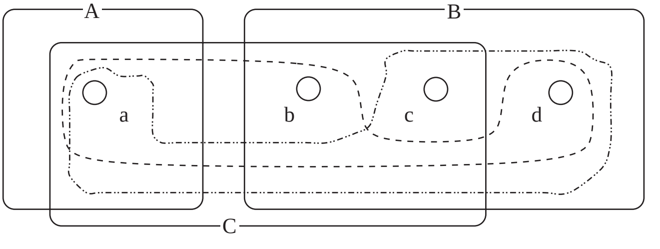
Subcase (1.2): $|B| \geqslant 6$.
We claim that there exists $C_0 \in$ *Supp* $P(\mathbf{x})$ such that $|C_0| = 2$. We know that there exists $C \in$ *Supp* $P(\mathbf{x})$ different from $\emptyset$ and E. If $|C| > 2$, then consider *Supp* $P^{\backslash(E\backslash C)}(\mathbf{x})$. Clearly, $\emptyset, C \in$ *Supp* $P(\mathbf{x})$ and thus $(\emptyset, C)$ satisfies the 2-step axiom. Therefore, *Supp* $P^{\backslash(E\backslash C)}(\mathbf{x})$ contains a 2-subset C_0 and so does *Supp* $P(\mathbf{x})$. Now, consider $P^{/C_0}(\mathbf{x})$. Since $\emptyset, E \setminus C_0 \in$ *Supp* $P^{/C_0}(\mathbf{x})$, for any $b_i \in E\backslash C_0$, there exists a 2-subset C_i in *Supp* $P^{/C_0}(\mathbf{x})$ which contain b_i. Hence, $C_0 \cup C_i \in$ *Supp* $P(\mathbf{x})$. Since $C_0 \cup C_i \subsetneq E$, $(\emptyset, C_0 \cup C_i)$ satisfies the 2-step axiom in $P^{\backslash(E\backslash C_0\backslash C_i)}(\mathbf{x})$. By (1.1), b_i is contained in a 2-subset, say C', in *Supp* $P^{\backslash(E\backslash C_0\backslash C_i)}(\mathbf{x})$. Hence, we showed that for any $b_i \in E \setminus C_0$, there exists a 2-subset $C' \in$ *Supp* $P(\mathbf{x})$ that contains b_i. Therefore, (A, B) satisfies the 2-step axiom.

Subcase 2: $|A| = 1$ and $|B| \geqslant 3$.
Subcase (2.1) $|B| = 3$.



Let $A := \{a\}$ and $B := \{b, c, d\}$. By Corollary 18, there exists $C \in \mathcal{C}[A, B]$ and we may assume that $C := \{a, b, c\}$. To show that (A, B) satisfies the 2-step axiom, we consider the following two cases which are part (i) and (ii) of the 2-step axiom in Lemma 19.

(i) For $a \in A$.



We need to show that there exists $b' \in B$ such that $A \setminus a \cup b' = \{b'\} \in Supp\, P(\mathbf{x})$. Suppose that there is no singleton in *Supp* $P(\mathbf{x})$ except for $A = \{a\}$. We will show that we can always find a solution $\mathbf{x}^* := (x_a^*, x_b^*, x_c^*, x_d^*) \in \mathbb{C}^E$ to the equation $P(\mathbf{x}) = 0$ such that $\operatorname{Re} x_e > 0$ for any $e \in E$, which then will contradict the assumption that $P(\mathbf{x})$ has the HPP. So, we may assume

$$P(\mathbf{x}) = x_a + Kx_bx_cx_d + Lx_ax_bx_c + Mx_ax_bs_d + Nx_ax_cx_d, \tag{4}$$

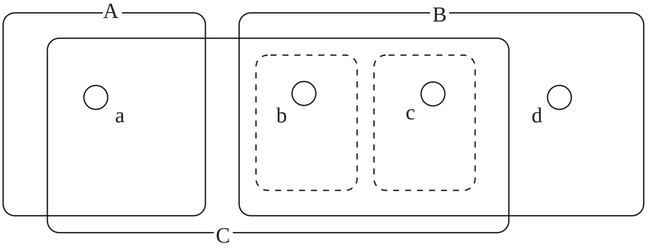
where K, L, M and N are nonnegative real numbers (Note that $K \neq 0$ and $L \neq 0$.)

Letting $R := \max\{L, M, N, 1\}$, we define $x^* \in \mathbb{C}^E$ as follows:

$$x_b^* = x_c^* = x_d^* = (1/\sqrt{6R}\,)e^{\pi i/4},$$
$$x_a^* = P_1(x_b^*, x_c^*, x_d^*) := \frac{Kx_b^*x_c^*x_d^*}{1 + Lx_b^*x_c^* + Mx_b^*x_d^* + Nx_c^*x_d^*}.$$

Clearly, we have $\operatorname{Re} \mathbf{x}_e^* > 0$ and $P(\mathbf{x}^*) = 0$. This implies that $P(\mathbf{x})$ does not have the HPP, which contradicts our assumption. Hence, there exists a singleton $\{e\} \in Supp\, P(\mathbf{x})$ for some $e \in B$ and thus (A, B) satisfies the 2-step axiom for $a \in A$ in $P(\mathbf{x})$.

(ii) For $e \in B$.



When $e \in \{b, c\}$, then there exists $f \in \{b, c\} \setminus e$ such that $A \cup e \cup f = \{a, b, c\} = C \in \mathit{Supp}\, P(\mathbf{x})$. Thus it remains to show that for d, there exists either a singleton $\{d\}$ or a 3-element set $\{a, d, g\}$ in $\mathit{Supp}\, P(\mathbf{x})$ for some $g \in B \setminus d$ $(= \{b, c\})$. Suppose that no set in $\mathit{Supp}\, P(\mathbf{x})$ contains d except for B. Then we may assume

$$P(\mathbf{x}) = x_a + Kx_a x_b x_c + Lx_b + Mx_c + Nx_b x_c x_d, \tag{5}$$

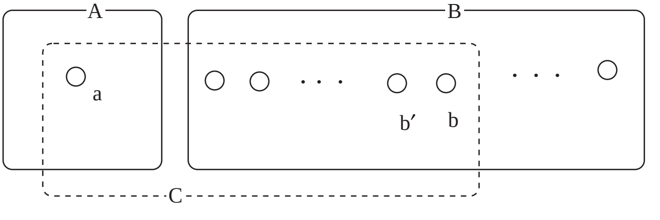
where $K,\ L,\ M$, and N are nonnegative real coefficients and $KN \neq 0$ and $L + M \neq 0$ (by (i)). Let $R = \max\{1, L^2, M^2, K\}$. We define $\mathbf{x}^* \in \mathbb{C}^E$ as follows:

$$\begin{aligned}
&x_b^* = x_c^* = (1/\sqrt{2R})\, e^{\pi i/3},\\
&x_d^* = (4\sqrt{2}\, R/N)\, e^{i\pi/3},\\
&x_a^* = P_2(x_b^*, x_c^*, x_d^*) := \frac{Lx_b^* + Mx_c^* + Nx_b^* x_c^* x_d^*}{1 + Kx_b^* x_c^*}.
\end{aligned}$$

Clearly, we have $P(\mathbf{x}^*) = 0$ and $\mathrm{Re}\, \mathbf{x}_e^* > 0$. Therefore $\{d\}$ or $\{a, d, g\}$ is in $\mathit{Supp}\, P(\mathbf{x})$.

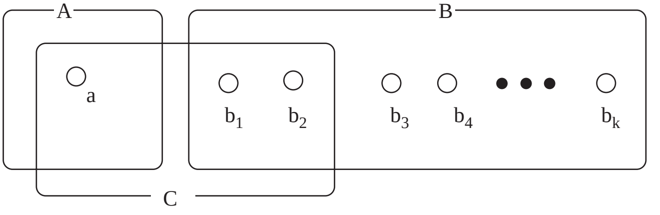
From (i) and (ii), we proved that (A, B) satisfies the 2-step axiom in $P(\mathbf{x})$.

Subcase (2.2): $|B| \geqslant 5$.



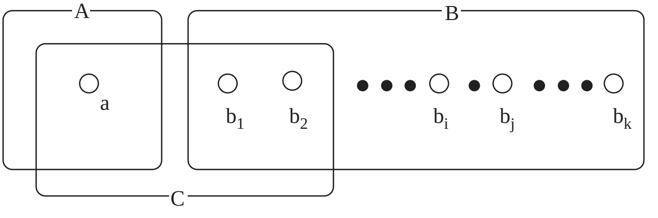
Let $A = \{a\}$ and $B = \{b_1, b_2, \ldots, b_k\}$ for some odd number $k \geqslant 5$. By Lemma 18, there exists $C \in \mathcal{C}[A, B]$. Choose C in $\mathcal{C}[A, B]$ such that $|C \cap B|$ is minimum. We claim that $|C \cap B| = 2$. Note that $A \cap C = A$ and $C \cap B \subsetneqq B$ because $|C|$ is odd. Suppose $|C| > 3$ and consider $A \setminus a$, $C \setminus a$ in $\mathit{Supp}\, P^{/a}(\mathbf{x})$. By the induction hypothesis, $(C \setminus a, \emptyset)$ satisfies the 2-step axiom in $P^{/a}(\mathbf{x})$. Hence, for $b \in C \setminus a$, there exists $b' \in (C \setminus a) \setminus b$ such that $(C \setminus a) \setminus b \setminus b' \in \mathit{Supp}\, P^{/a}(\mathbf{x})$, i.e., $C \setminus b \setminus b' \in \mathit{Supp}\, P(\mathbf{x})$. Denote $C \setminus b \setminus b'$ by C'. Then $C' \in \mathcal{C}[A, B]$ and $C' \subsetneqq C$, i.e., $|C' \cap B| < |C \cap B|$. This is a contradiction. Therefore, $|C| = 3$ and thus we may assume $C := \{a, b_1, b_2\}$. Now we will show that (A, B) satisfies the 2-step axiom.

(i) For a.



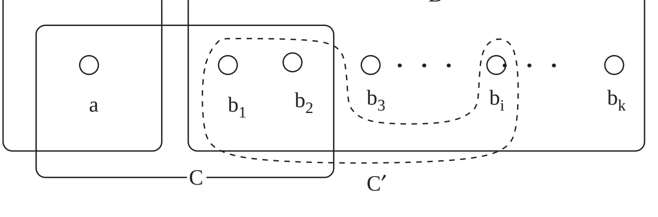
Consider C, B and $P^{/C\cap B}(\mathbf{x})$. Then both $C \setminus B(= A)$ and $B \setminus C$ are in *Supp* $P^{/C\cap B}(\mathbf{x})$. By induction hypothesis, $(A, B \setminus C)$ satisfies the 2-step axiom for a in $P^{/C\cap B}(\mathbf{x})$. Thus there exists $b \in B \setminus C$ such that $A \setminus a \cup b = \{b\} \in$ *Supp* $P^{/C\cap B}(\mathbf{x})$ i.e., $C' := C \setminus a \cup b \in$ *Supp* $P(\mathbf{x})$. Hence $C' = \{b_1, b_2, b\}$. Since $|C'| = 3 < |B|$, $B \setminus C' \neq \emptyset$. Now consider A, C' and $P^{\setminus(B\setminus C')}(\mathbf{x})$. Then $A, C' \in$ *Supp* $P^{\setminus(B\setminus C')}(\mathbf{x})$. By Subcase (2.1), there exists a singleton, say, $\{c'\} \subset C'$ in *Supp* $P^{\setminus(B\setminus C')}(\mathbf{x})$. Therefore $\{c'\} \in$ *Supp* $P(\mathbf{x})$ and this implies that (A, B) satisfies the 2-step axiom for a in $P(\mathbf{x})$.

(ii) For $b_i \in B$, $i = 1, 2, \ldots, k$.



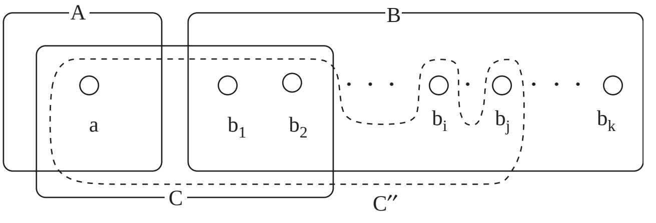
The existence of C' and Lemma 20 satisfy the 2-step axiom of (A, B) for b_1 and b_2 in $P(\mathbf{x})$. Hence we may assume $i \geqslant 3$ in b_i. Then $b_i \in B \setminus C$. Consider C, B and *Supp* $P^{/C\cap B}(\mathbf{x})$. Then $(C \setminus B, B \setminus C)$ satisfies the 2-step axiom in $P^{/C\cap B}(\mathbf{x})$, i.e., for b_i, *Supp* $P^{/C\cap B}(\mathbf{x})$ contains $(C \setminus B) \cup b_i \setminus a$ or $(C \setminus B) \cup b_i \cup b_j$ for some $j \in \{3, 4, \ldots, k\} \setminus i$. Hence, we have $C_i' := C \cup b_i \setminus a \in$ *Supp* $P(\mathbf{x})$ or $C_i'' := C \cup b_i \cup b_j \in$ *Supp* $P(\mathbf{x})$ for every $i \geqslant 3$.

(a) If $C_i' \in$ *Supp* $P(\mathbf{x})$,



then $A, C_i' \in$ *Supp* $P^{\setminus(B\setminus C_i')}(\mathbf{x})$. By Subcase (2.2) (ii), *Supp* $P^{\setminus(B\setminus C_i')}$ contains $\{b_i\}$ or $\{a, b_l, b_i\}$ for some $l \in \{1, 2\}$ and so does *Supp* $P(\mathbf{x})$.

(b) If $C_i'' \in Supp\, P(\mathbf{x})$,

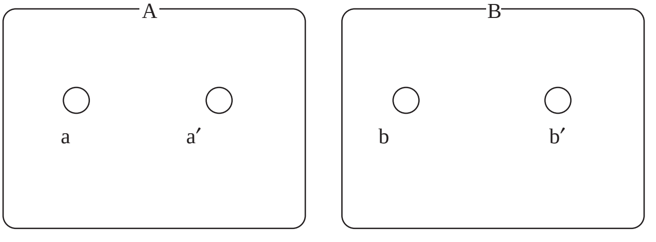


then since $C_i'' \in \mathcal{C}[A, B]$ and $b_i \in C_i''$, Lemma 20(i) applies. Hence (A, B) satisfies the 2-step axiom for b_i in $P(\mathbf{x})$.

Therefore, by (a) and (b), (A, B) satisfies the 2-step axiom for b_i in $P(\mathbf{x})$.

Subcase 3: $|A| = 2$ and $|B| \geqslant 2$.

Subcase (3.1) $|B| = 2$.



Let $A := \{a, a'\}$ and $B := \{b, b'\}$. If $\emptyset \in Supp\, P(\mathbf{x})$ then (A, B) satisfies the 2-step axiom for any $e \in A$. If $E \in Supp\, P(\mathbf{x})$ then (A, B) has the 2-step axiom for any $f \in B$. We consider four possible cases with respect to the existence of $\emptyset$ and E in $Supp\, P(\mathbf{x})$.

(1) $\emptyset \in Supp\, P(\mathbf{x})$ and $E \in Supp\, P(\mathbf{x})$.

By the above argument (A, B) satisfies the 2-step axiom in $P(\mathbf{x})$.

(2) $\emptyset \in Supp\, P(\mathbf{x})$ and $E \notin Supp\, P(\mathbf{x})$.

Since $\emptyset \in Supp\, P(\mathbf{x})$, the 2-step axiom holds for a and a'. By Lemma 18, there exists $C \in \mathcal{C}[A, B]$ and $C \neq E$. So we may assume $C = \{a', b\}$. Since $C = A \cup b \setminus a$, (A, B) satisfies the 2-step axiom for b and thus we only need to show the 2-step axiom for b'. Suppose neither $\{a, b'\}$ nor $\{a', b'\}$ is in $Supp\, P(x)$. Then we may assume

$$P(\mathbf{x}) = 1 + Kx_a x_{a'} + Lx_b x_{b'} + Mx_a x_b + Nx_{a'} x_b, \tag{6}$$

where K, L, M and N are nonnegative real numbers and $KLN \neq 0$.

Assign values to x_a^* and $x_{a'}^*$ as

$$x_a^* = x_{a'}^* = \sqrt{(\sqrt{3} + 3)/(\sqrt{2}K)}\, e^{3\pi i/8}.$$

We have

$$\begin{aligned} Mx_a^* + Nx_{a'}^* &= (M+N)\sqrt{(\sqrt{3}+3)/(\sqrt{2}K)}\, e^{3\pi i/8} \\ &= r + si \end{aligned}$$

for some positive real numbers r and s.
Now we assign to $x_{b'}^*$ a positive real number as follows:

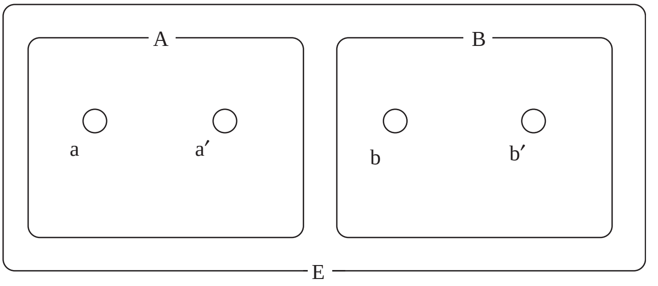
$$x_{b'}^* = (2\sqrt{3}s - r)/L.$$

We define x_b^* as

$$x_b^* = P_3(x_a^*, x_b^*, x_{b'}^*) := \frac{1 + Kx_a^* x_{a'}^*}{Lx_{b'}^* + Mx_a^* + Nx_{a'}^*}.$$

Since $P(\mathbf{x}^*) = 0$ and $\mathrm{Re}\,\mathbf{x}^* > 0$, $P(\mathbf{x})$ does not satisfy the HPP in contradiction to the assumption. Therefore, *Supp* $P(\mathbf{x})$ contains $\{a, b'\}$ or $\{a', b'\}$ and thus the 2-step axiom holds for (A, B) in $P(\mathbf{x})$.

(3) $\emptyset \notin$ *Supp* $P(\mathbf{x})$ and $E \in$ *Supp* $P(\mathbf{x})$.
Since $E \in$ *Supp* $P(\mathbf{x})$, we only need to show the 2-step axiom for $e \in A$.



First, we show that there exist more than three sets in *Supp* $P(\mathbf{x})$.
Suppose that

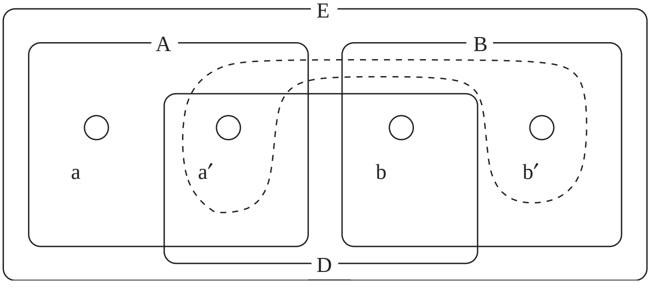
$$P(\mathbf{x}) = x_a x_{a'} x_b x_{b'} + Lx_a x_{a'} + Mx_b x_{b'} \tag{7}$$

for some positive real numbers L, M.
We define $\mathbf{x}^* \in \mathbb{C}^E$ by

$$\begin{aligned} x_a^* = x_{a'}^* &= \sqrt[4]{2M^2}e^{3\pi i/8}, \\ x_b^* = x_{b'}^* &= \sqrt[4]{2L^2}e^{-3\pi i/8}. \end{aligned}$$

Then $\mathrm{Re}\,\mathbf{x}^* > 0$ and $P(\mathbf{x}^*) = 0$. Therefore, there must exist another set D in *Supp* $P(\mathbf{x}) \setminus \{A, B, C\}$.

We may assume $D := \{a', b\}$.



D approves the 2-step axiom for a. So it remains to show the axiom for a'. Since $\emptyset \notin$ *Supp* $P(\mathbf{x})$, we need to prove that there exists either $\{a, b\}$ or $\{a, b'\}$ in *Supp* $P(\mathbf{x})$. Assume that none of them exist in *Supp* $P(\mathbf{x})$. Then we may assume

$$P(\mathbf{x}) = x_a x_{a'} x_b x_{b'} + K x_a x_{a'} + L x_b x_{b'} + M x_{a'} x_b + N x_{a'} x_{b'}, \tag{8}$$

where K, L, M and N are nonnegative real coefficients and $KLM \neq 0$.
First we define x_b^* and $x_{b'}^*$ as follows:

$$x_b^* = x_{b'}^* = \sqrt[4]{(K^2/2)}\, e^{-3\pi i/8}.$$

Then,

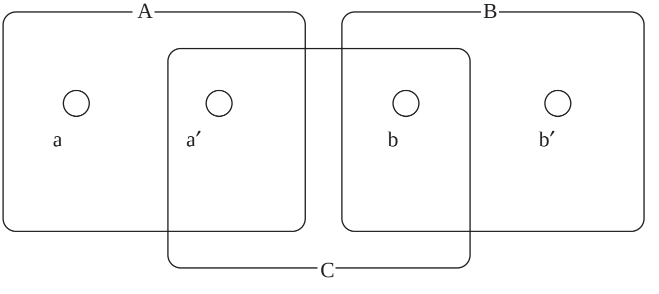
$$M/x_{b'}^* + N/{x_b}^* = (M+N)\sqrt[4]{(2/K^2)}\, e^{3\pi i/8} = r + is$$

for some $r, s > 0$.
Now we define x_a^* and $x_{a'}^*$:

$$x_a^* = \sqrt{2} s e^{i\pi/4},$$
$$x_{a'}^* = L/((s-r) - 2si).$$

We have $\mathrm{Re}\,\mathbf{x}^* > 0$ and $P(\mathbf{x}^*) = 0$ and this contradicts that $P(\mathbf{x})$ has the HPP. Thus, there exists $\{a, b\}$ or $\{a, b'\}$ in *Supp* $P(\mathbf{x})$. Therefore, the 2-step axiom holds for all $e \in A$ and $f \in B$ in $P(\mathbf{x})$.

(4) $\emptyset \notin$ *Supp* $P(\mathbf{x})$ and $E \notin$ *Supp* $P(\mathbf{x})$.



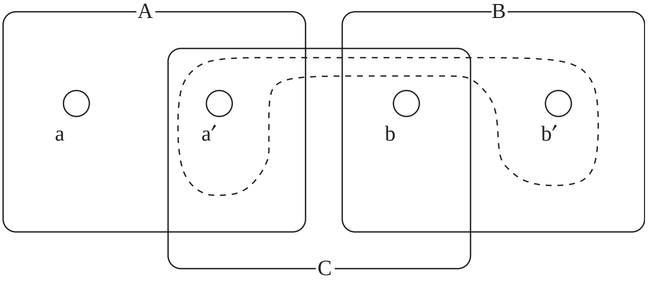
By Corollary 18, there exists a 2-element set $C \in \mathcal{C}[A, B]$. We may assume that $C := \{a', b\}$. Suppose that there are only 3 elements A, B and C in *Supp* $P(\mathbf{x})$. Then

$$P(\mathbf{x}) = x_a x_{a'} + L x_{a'} x_b + M x_b x_{b'}. \tag{9}$$

for some positive real numbers L and M.
Define $x^* \in \mathbb{C}^E$ as follows:

$$x_a^* = \frac{1}{2} cos\frac{\pi}{8} + i\left(\frac{5}{2} sin\frac{\pi}{8}\right),$$
$$x_{a'}^* = \frac{2}{3} e^{3\pi i/8},$$
$$x_b^* = e^{-i\pi/8}/L,$$
$$x_{b'}^* = (L\, e^{-3\pi i/8})/M.$$

Again, we have $\operatorname{Re} \mathbf{x}_e^* > 0$ and $P(x^*) = x_{a'}^*(x_a^* + L x_b^*) + M x_b^* x_{b'}^* = 0$, which is a contradiction. Therefore, there must exist some other set, say $D \in$ *Supp* $P(\mathbf{x})$. If $D = \{a, b'\}$ then (A, B) satisfies the 2-step axiom in $P(\mathbf{x})$. So suppose $\{a, b'\} \notin$ *Supp* $P(\mathbf{x})$ and $D = \{a', b'\} \in$ *Supp* $P(\mathbf{x})$. Then, (A, B) satisfies the 2-step axiom for all elements of E except for a'.



Assume there exists neither $\{a, b\}$ nor $\{a, b'\}$ in *Supp* $P(\mathbf{x})$. Then

$$P(\mathbf{x}) = x_a x_{a'} + L x_{a'} x_b + M x_b x_{b'} + N x_{a'} x_{b'}, \tag{10}$$

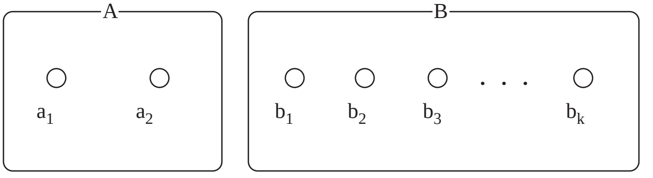
where L, M and N are positive real numbers.
Define $\mathbf{x}^* \in \mathbb{C}^E$ as follows:

$$x_a^* = r + 3ri,$$
$$x_{a'}^* = \frac{1}{2\sqrt{2}\,r} e^{i\pi/4},$$
$$x_{b'}^* = e^{-i\pi/4}/\sqrt{M},$$
$$x_b^* = e^{-i\pi/4}/\sqrt{M},$$

where $r = (L + N)/\sqrt{2M}$. Then $\operatorname{Re} \mathbf{x}^* > 0$ and we have $P(\mathbf{x}^*) = 0$. Therefore there must exist $\{a, b\}$ or $\{a, b'\}$ in *Supp* $P(\mathbf{x})$ and hence the 2-step axiom holds for (A, B). By symmetry the case when $D = \{a, b\}$ and there exists no 2-element set containing b' except for B will result in $P(x^*) = 0$ for some $x^* \in \mathbb{C}^E$ with $\operatorname{Re} x^* > 0$.

Therefore, (A, B) satisfies the 2-step axiom in $P(\mathbf{x})$ in any of the above cases from (1) to (4).

Subcase (3.2): $|B| \geqslant 4$.



Let $A = \{a_1, a_2\}$ and $B = \{b_1, b_2, b_3, \ldots, b_k\}$ for some even number $k \geqslant 4$. Then by Corollary 18, there exists $C \in \mathcal{C}[A, B]$. In particular, we pick an element $C \in \mathcal{C}[A, B]$ so that $|A \cap C| = \max\{|A \cap D| : D \in \mathcal{C}\}$.

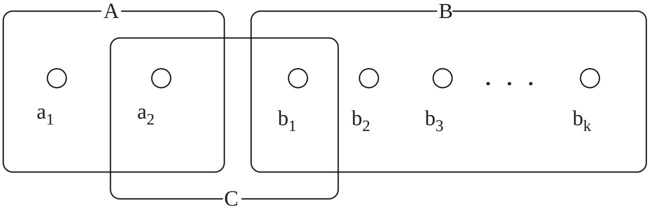
(1) $|A \cap C| = 1$.

We claim that $|C \cap B| = 1$.

Suppose that $|C \cap B| > 1$. Then since $|A \cap C| = 1$, we must have $|C \cap B| \geqslant 3$. Without loss of generality, we may assume that $\{a_2, b_1, b_2, b_3\} \subseteq C$. Consider A, C and $P^{/a_2}(\mathbf{x})$. By induction hypothesis, $(C \setminus a_2, A \setminus a_2)$ satisfies the 2-step axiom in $P^{/a_2}(\mathbf{x})$, especially for $a_1 \in A \setminus a_2$. Hence, there exists $b_i \in C \setminus A$ such that $(C \setminus a_2) \cup a_1 \setminus b_i \in$ *Supp* $P^{/a_2}(\mathbf{x})$ and hence $C' := C \cup a_1 \setminus b_i \in$ *Supp* $P(\mathbf{x})$. Note that $C' \in \mathcal{C}[A, B]$ and $|A \cap C'| = 2 > 1 = |A \cap C|$. This contradicts that $|C \cap A| = 1$ is maximum. Therefore, we must have $|C \cap B| = 1$.

We may now assume that $C = \{a_2, b_1\}$.

(i) For $a \in A$.

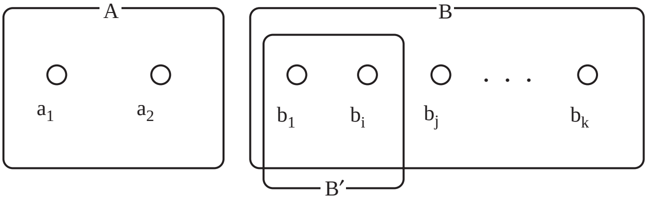


For $a = a_1$, we have $C = A \setminus a_1 \cup b_1 \in$ *Supp* $(\mathbf{x})$. Hence it remains to show the 2-step axiom for $a = a_2$. Consider C, B and $P^{/b_1}(\mathbf{x})$. For $a_2 \in C \setminus b_1$, $(C \setminus b_1, B \setminus b_1)$ satisfies the 2-step axiom in $P^{/b_1}(\mathbf{x})$, that is, there exists $b_j \in B \setminus b_1$ such that $(C \setminus b_1) \setminus a_2 \cup b_j \in$ *Supp* $P^{/b_1}(\mathbf{x})$. Hence $C' := C \setminus a_2 \cup b_j = \{b_1, b_j\} \in$ *Supp* $P(\mathbf{x})$ for some $2 \leqslant j \leqslant k$. Since $|B| > 2$, $B \setminus C' \neq \emptyset$. Now consider A and C' in *Supp* $P^{\setminus(B \setminus C')}(\mathbf{x})$. Then by Subcase (3.1), (A, C') satisfies the 2-step axiom for a_2 in $P^{\setminus(B \setminus C')}(\mathbf{x})$. Hence, we have $\emptyset \in$ *Supp* $P(\mathbf{x})$ or $A \setminus a_2 \cup b' \in$ *Supp* $P(\mathbf{x})$ for some $b' \in C' (\subset B)$. Therefore (A, B) satisfies the 2-step axiom for both a_1 and a_2 in $P(\mathbf{x})$.

(ii) For b_i, $i \in \{1, 2, \ldots, k\}$.

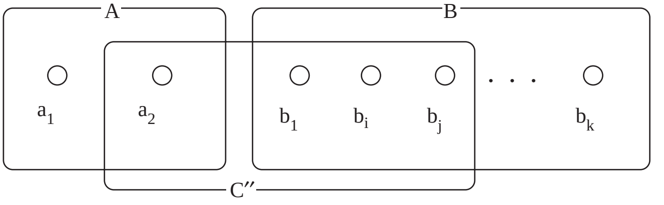
For b_1 we have $C = A \cup b_1 \setminus a_1 \in$ *Supp* $P(\mathbf{x})$. So we may assume $i \in \{2, 3, \ldots, k\}$. Consider $C \setminus b_1$, $B \setminus b_1 \in$ *Supp* $P^{/b_1}(\mathbf{x})$. By induction hypothesis, $(C \setminus b_1, B \setminus b_1)$ satisfies the 2-step axiom for $b_i \in B \setminus b_1$ in $P^{/b_1}(\mathbf{x})$. Hence, we have $(C \setminus b_1) \setminus a_2 \cup b_i \in \mathit{Supp}\, P^{/b_1}(\mathbf{x})$ or $(C \setminus b_1) \cup b_i \cup b_j \in \mathit{Supp}\, P^{/b_1}(\mathbf{x})$ for some $j \in \{2, 3, \ldots, k\} \setminus i$. Thus $\mathit{Supp}\, P(\mathbf{x})$ contains $B' := \{b_1, b_i\}$ or $C'' := \{a_2, b_1, b_i, b_j\}$.

(a) When $B' \in \mathit{Supp}\, P(\mathbf{x})$,



we consider $P^{\setminus(B\setminus B')}(\mathbf{x})$ and (A, B'). By Subcase (3.1), (A, B') satisfies the 2-step axiom for b_i in $P^{\setminus(B\setminus B')}(\mathbf{x})$. Since $B' \subset B$ and $\mathit{Supp}\, P^{\setminus(B\setminus B')}(\mathbf{x}) \subset \mathit{Supp}\, P(\mathbf{x})$, (A, B) also satisfies the 2-step axiom for b_i in $P(\mathbf{x})$.

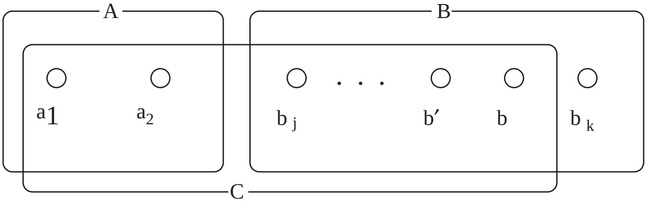
(b) When $C'' \in \mathit{Supp}\, P(\mathbf{x})$,



$C'' \in \mathcal{C}[A, B]$ and $b_i \in C''$. Hence by Lemma 20, (A, B) satisfies the 2-step axiom for b_i in $P(\mathbf{x})$.

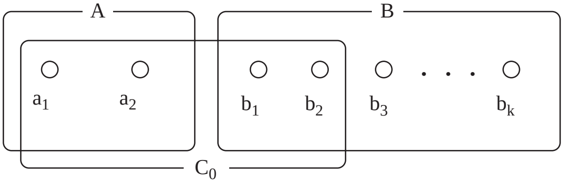
Therefore, for any b_i, (A, B) satisfies the 2-step axiom in $P(\mathbf{x})$.

(2) $|A \cap C| = 2$.



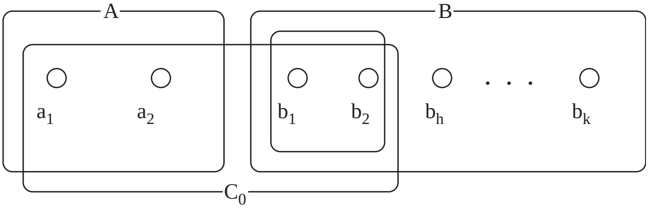
We claim that there exists $C_0 \in \mathcal{C}[A, B]$ such that $|C_0 \cap A| = 2$ and $|C_0 \cap B| = 2$. Note that $|C \cap B|$ is an even number. Hence, we choose $C_0 \in \mathcal{C}[A, B]$ so that $|C_0 \cap A| = 2$ and $|C_0 \cap B| = 2M$ is as small as possible. Consider $A \setminus A\ (= \emptyset)$ and $C_0 \setminus A$ in $\mathit{Supp}\, P^{/A}(\mathbf{x})$. By induction hypothesis, $(C_0 \setminus A, \emptyset)$ satisfies the 2-step axiom, in particular, for any $b \in C_0 \setminus A = C_0 \cap B$. Therefore, there exists $b' \in C_0 \setminus A \setminus b$ such that $(C_0 \setminus A) \setminus b \setminus b' \in \mathit{Supp}\, P^{/A}(\mathbf{x})$, i.e., $C_1 := C_0 \setminus b \setminus b' \in \mathit{Supp}\, P(\mathbf{x})$. But since $|C_0 \cap B| = 2M \geqslant 4$, we have $|C_1 \cap B| = |C_0 \cap B| - 2 \geqslant 2$ and thus $C_1 \cap B \neq \emptyset$. Hence, $C_1 \in \mathcal{C}[A, B]$ and moreover $|C_1 \cap A| = 2$ and $|C_1 \cap B| = 2M - 2$, which contradicts that $|C_0 \cap B| = 2M$ is minimum. Therefore, $|C_0 \cap B| = 2$ and we may assume that $C_0 = \{a_1, a_2, b_1, b_2\}$.

(i) For a_i, $i = 1, 2$.



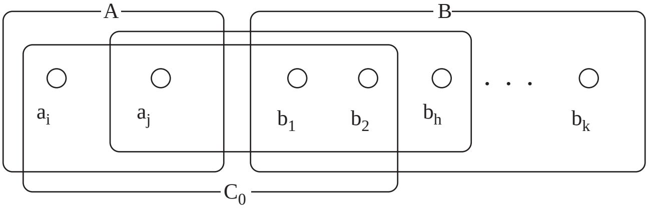
Consider $C_0 \setminus \{b_1, b_2\} := A$ and $B \setminus \{b_1, b_2\}$ in *Supp* $P^{/\{b_1,b_2\}}(\mathbf{x})$. Then for a_i, we have $A \setminus a_i \setminus a_j = \emptyset \in$ *Supp* $P^{/\{b_1,b_2\}}(\mathbf{x})$ for $j \in \{1, 2\} \setminus i$ or $A \setminus a_i \cup b_h \in$ *Supp* $P^{/\{b_1,b_2\}}(\mathbf{x})$ for some $3 \leqslant h \leqslant k$. Hence we have $\{b_1, b_2\} \in$ *Supp* $P(\mathbf{x})$ or $\{a_j, b_1, b_2, b_h\} \in$ *Supp* $P(\mathbf{x})$.

(a) When $\{b_1, b_2\} \in$ *Supp* $P(\mathbf{x})$,



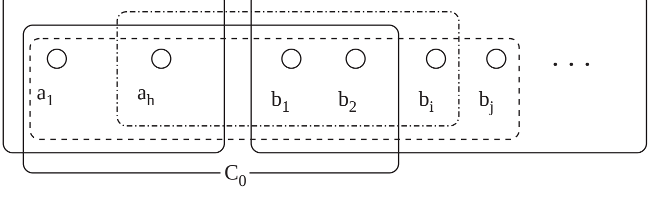
$(A, \{b_1, b_2\})$ satisfies the 2-step axiom in $P^{\setminus(B\setminus\{b_1,b_2\})}(\mathbf{x})$ by Subcase (3.1). Since $\{b_1, b_2\} \subset B$ and *Supp* $P^{\setminus(B\setminus\{b_1,b_2\})}(\mathbf{x}) \subset$ *Supp* $P(\mathbf{x})$, (A, B) satisfies the 2-step axiom for a_i in $P(\mathbf{x})$.

(b) When $\{a_j, b_1, b_2, b_h\} \in$ *Supp* $P(\mathbf{x})$,



Lemma 20 applies to $C_0' := \{a_j, b_1, b_2, b_h\} \in \mathcal{C}[A, B]$ and $a_i \notin C_0'$ in $P(\mathbf{x})$. Therefore, (A, B) satisfies the 2-step axiom for any $a_i \in A$ in $P(\mathbf{x})$.

(ii) For b_i, $i = 1, 2, \ldots, k$.

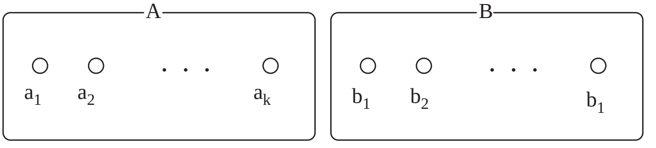


By Lemma 20, (A, B) satisfies the 2-step axiom for b_1 and b_2 in $P(\mathbf{x})$. Thus we may assume $i = 3, \ldots, k$.

Again, consider $C_0 \setminus \{b_1, b_2\}(= A)$ and $B \setminus \{b_1, b_2\}$ in *Supp* $P^{/\{b_1,b_2\}}(\mathbf{x})$. Then, for b_i we have $A \cup b_i \setminus a_l \in$ *Supp* $P^{/\{b_1,b_2\}}(\mathbf{x})$ for some $l \in \{1, 2\}$ or $A \cup b_i \cup b_j \in$ *Supp* $P^{/\{b_1,b_2\}}(\mathbf{x})$ for some $j \in \{3, 4, \ldots, k\} \setminus i$. Hence *Supp* $P(\mathbf{x})$ contains $C_1 := C_0 \cup b_i \setminus a_l$ or $C_2 := C_0 \cup b_i \cup b_j$. Then both C_1 and C_2 are in $\mathcal{C}[A, B]$ and contain b_i. Hence by Lemma 20, (A, B) satisfies the 2-step axiom for b_i in $P(\mathbf{x})$.

Therefore, when $|A| = 2$ and $|B| \geqslant 4$, (A, B) satisfies the 2-step axiom in $P(\mathbf{x})$.

Subcase 4: $|A| \geqslant 3$ and $|B| \geqslant 3$.

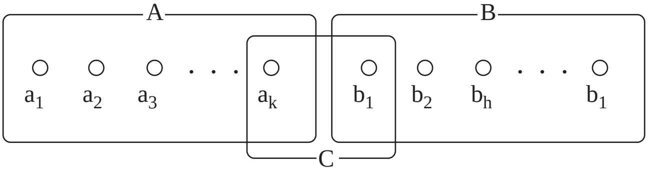


Let $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_l\}$ for some $k, l \geqslant 3$. Then by Corollary 18, there exists $C \in \mathcal{C}[A, B]$. Again, we pick $C \in \mathcal{C}$ so that $|A \cap C| = \max\{|A \cap D| : D \in \mathcal{C}[A, B]\}$.

(1) $|A \cap C| = 1$.

By the same argument as we made in Subcase (3.2)(1), we have $|C \cap B| = 1$. So we may assume $C := \{a_k, b_1\}$. Hence this case happens only for even polynomials. Let us show that (A, B) satisfies the 2-step axiom for all $a \in A$ and $b \in B$.

(i) For a_i, $i = 1, 2, \ldots, k$.



Since $C \in \mathcal{C}[A, B]$ and $a_i \notin C$ for all $i \leqslant k - 1$, by Lemma 20, (A, B) satisfies the 2-step axiom in $P(\mathbf{x})$ for any a_i, $i \neq k$. Hence, it remains to show the 2-step axiom for a_k. Consider $A \setminus a_k$, $C \setminus a_k$ in *Supp* $P^{/a_k}(\mathbf{x})$. By induction hypothesis, $(C \setminus a_k, A \setminus a_k)$ satisfies the 2-step axiom, especially for any a_i, $i \leqslant k - 1$. Then *Supp* $P^{/a_k}(\mathbf{x})$ contains $(C \setminus a_k) \cup a_1 \cup a_j$ for some $2 \leqslant j \leqslant k - 1$ or $(C \setminus a_k) \cup a_1 \setminus b_1 = \{a_1\}$, i.e., *Supp* $P(\mathbf{x})$ contains $C \cup a_1 \cup a_j$ or $\{a_1, a_k\}$. But the former case is not possible because $C \cup a_1 \cup a_j \in \mathcal{C}[A, B]$ and $|(C \cup a_1 \cup a_j) \cap A| = 3 > 1$. Therefore, there must exist $C' := \{a_1, a_k\}$. Now we consider C' and B in *Supp* $P^{\setminus(A \setminus C')}(\mathbf{x})$. Then for a_k, (C', B) satisfies the 2-step axiom in $P^{\setminus(A \setminus C')}(\mathbf{x})$, that is, *Supp* $P^{\setminus(A \setminus C')}(\mathbf{x})$ has $C' \setminus a_k \setminus a_1 = \emptyset$ or $C' \setminus a_k \cup b'$ for some $b' \in B$. Hence *Supp* $P(\mathbf{x})$ contains $\emptyset$ or $C'' := \{a_1, b'\}$.

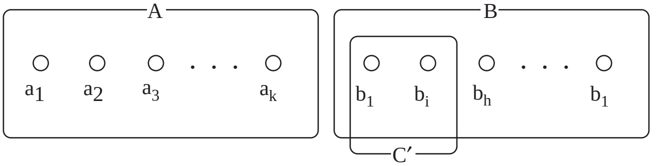
(a) If $\emptyset \in$ *Supp* $P(\mathbf{x})$, then in *Supp* $P^{\setminus B}(\mathbf{x})$, $(A, \emptyset)$ satisfies the 2-step axiom, especially for a_k. Hence, *Supp* $P^{\setminus B}(\mathbf{x})$ contains $A \setminus a_k \setminus a'$ for some $a' \in A \setminus a_k$ and so does *Supp* $P(\mathbf{x})$.
(b) If $C'' = \{a_1, b'\} \in$ *Supp* $P(\mathbf{x})$, then $C'' \in \mathcal{C}[A, B]$ and $a_k \notin c''$. Thus, by Lemma 20, (A, B) satisfies the 2-step axiom for a_k in $P(\mathbf{x})$.

Therefore in both cases (a) and (b), (A, B) satisfies the 2-step axiom for a_k and thus for every a_i, $i = 1, 2, \ldots, k$.

(ii) For b_i, $i = 1, 2, \ldots, l$.

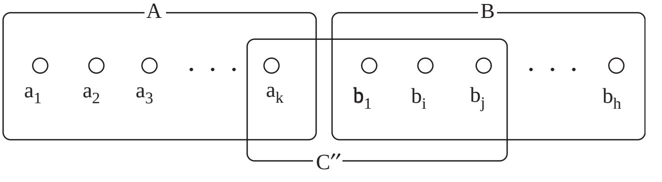
Since $C \in \mathcal{C}[A, B]$ and $b_1 \in C$, by Lemma 20, (A, B) satisfies the 2-step axiom for b_1 in $P(\mathbf{x})$. So we may assume that $i \geqslant 2$. Now let us consider $C \setminus b_1 = \{a_k\}$, $B \setminus b_1$ and $P^{/b_1}(\mathbf{x})$. Then, $(\{a_k\}, B \setminus b_1)$ satisfies the 2-step axiom in $P^{/b_1}(\mathbf{x})$ for any b_i, $i \geqslant 2$. This implies that *Supp* $P^{/b_1}(\mathbf{x})$ contains $\{b_i\}$ or $\{a_k, b_i, b_j\}$ for some $j \in \{2, 3, \ldots, l\} \setminus i$. Hence *Supp* $P(\mathbf{x})$ contains $C' := \{b_1, b_i\}$ or $C'' := \{a_k, b_1, b_i, b_j\}$.

(a) If $C' \in$ *Supp* $P(\mathbf{x})$,



then we consider (A, C') in $P^{\setminus(B \setminus C')}(\mathbf{x})$. By induction hypothesis, (A, C') satisfies the 2-step axiom for $b_i \in C'$ in $P^{\setminus(B \setminus C')}(\mathbf{x})$. Since $C' \subset B$ and *Supp* $P^{\setminus(B \setminus C')}(\mathbf{x}) \subset$ *Supp* $P(\mathbf{x})$, (A, B) satisfies the 2-step axiom for b_i in $P(\mathbf{x})$.

(b) If $C'' \in$ *Supp* $P(\mathbf{x})$,



then $C'' \in \mathcal{C}[A, B]$ and $b_i \in C''$. Thus, by Lemma 20, (A, B) satisfies the 2-step axiom for b_i in $P(\mathbf{x})$.

Therefore (A, B) satisfies the 2-step axiom for any b_i, $i = 1, 2, \ldots, l$.

(2) $M_0 := |A \cap C| \geqslant 2$.

Pick a $C \in \mathcal{C}$ such that $|A \cap C| = M_0$ and $|C \cap B|$ is as small as possible. Then we claim that either $A \cap C = A$ or $|C \cap B| = 1$. Suppose $A \cap C \neq A$ and $|C \cap B| \geqslant 2$. Since $M_0 \geqslant 2$, we may assume that $a_1, a_2 \in A \cap C$ and $b_1, b_2 \in C \cap B$. Consider $(C \setminus A, A \setminus C)$ and $P^{/A \cap C}(\mathbf{x})$. Since $C \cap A \neq A$, there exists $a' \in C \setminus A$. Hence by the induction hypothesis, we have $(A \setminus C) \cup a' \cup a'' \in$ *Supp* $P^{/A \cap C}(\mathbf{x})$ for some $a'' \in A \setminus C \setminus a'$ or we have $(A \setminus C) \cup a' \setminus b_i \in$ *Supp* $P^{/A \cap C}(\mathbf{x})$ for some $b_i \in C \setminus A$. Thus $C' := A \cup a' \cup a'' \in$ *Supp* $P(\mathbf{x})$ or $C'' := A \cup a' \setminus b_i \in$ *Supp* $P(\mathbf{x})$. But then we have $C', C'' \in \mathcal{C}[A, B]$ and $|A \cap C'|, |A \cap C''| > M_0$, which is a contradiction. Therefore, either $A \cap C = A$ or $|C \cap B| = 1$ holds.

(i) For a_i, $i = 1, 2, \ldots, k$.

For $a \in A \setminus C$, (A, B) satisfies the 2-step axiom in $P(\mathbf{x})$ by Lemma 20. So let us assume $a_i \in A \cap C$. Consider C, B and $P^{/C \cap B}(\mathbf{x})$. Then $(C \setminus B, B \setminus C)$ satisfies the 2-step axiom in $P^{/C \cap B}(\mathbf{x})$, especially for $a_i \in C \setminus B = C \cap A$. Thus, we have $(C \setminus B) \setminus$

$a_i \setminus a_j \in Supp\, P^{/C\cap B}(\mathbf{x})$ for some $a_j \in (C \setminus B) \setminus a_i$ or we have $(C \setminus B) \setminus a_i \cup b' \in Supp\, P^{/C\cap B}(\mathbf{x})$ for some $b' \in B \setminus C$. Hence $C \setminus a_i \setminus a_j \in Supp\, P(\mathbf{x})$ or $C \setminus a_i \cup b' \in Supp\, P(\mathbf{x})$.

(a) When $|A \cap C| \geqslant 3$, if $C \setminus a_i \setminus a_j \in Supp\, P(\mathbf{x})$ then $(C \setminus a_i \setminus a_j) \cap A \neq \emptyset$ and thus $C \setminus a_i \setminus a_j \in \mathcal{C}[A, B]$. If $C \setminus a_i \cup b' \in Supp\, P(\mathbf{x})$, then also $C \setminus a_i \setminus a_j \in \mathcal{C}[A, B]$. Since $a_i \notin C \setminus a_i \setminus a_j$ and $a_i \notin C \setminus a_i \cup b'$, for both cases, Lemma 20 applies and hence (A, B) satisfies the 2-step axiom for a_i in $P(\mathbf{x})$.

(b) When $|A \cap C| = 2$, $A \cap C \neq A$ and thus we know that $|C \cap B| = 1$. If $C \setminus a_i \cup b' \in Supp\, P(\mathbf{x})$, then $C \setminus a_i \cup b' \in \mathcal{C}[A, B]$ and $a_i \notin C \setminus a_i \cup b'$. Hence by Lemma 20, (A, B) satisfies the 2-step axiom for a_i in $P(\mathbf{x})$. Now suppose $C \setminus a_i \cup b' \notin Supp\, P(\mathbf{x})$ and $C \setminus a_i \setminus a_j \in Supp\, P(\mathbf{x})$. We may assume $C \cap B = \{b_1\}$. Then consider $(A, \{b_1\})$ and $Supp\, P^{\setminus(B\setminus b_1)}(\mathbf{x})$. By induction hypothesis, $(A, \{b_1\})$ satisfies the 2-step axiom in $P^{\setminus(B\setminus b_1)}(\mathbf{x})$, in particular for a_i. Since $\{b_1\} \subset B$ and $Supp\, P^{\setminus(B\setminus b_1)}(\mathbf{x}) \subset Supp\, P(\mathbf{x})$, it follows that (A, B) satisfies the 2-step axiom for a_i in $P(\mathbf{x})$.

(ii) For b_i, $i = 1, 2, \ldots, l$.
By Lemma 20, (A, B) satisfies the 2-step axiom for $b \in C \cap B$ in $P(\mathbf{x})$. So we may assume $b_i \in B \setminus C$. Consider $(C \setminus B, B \setminus C)$ and $P^{/C\cap B}(\mathbf{x})$. For any $b_i \in B \setminus C$, $(C \setminus B, B \setminus C)$ satisfies the 2-step axiom in $P^{/C\cap B}(\mathbf{x})$. Hence, $(C \setminus B) \cup b_i \setminus a' \in Supp\, P^{/C\cap B}(\mathbf{x})$ for some $a' \in C \setminus B$ or $(C \setminus B) \cup b_i \cup b_j \in Supp\, P^{/C\cap B}(\mathbf{x})$ for some $b_j \in B \setminus C \setminus b_i$. This implies that $C \cup b_i \setminus a' \in Supp\, P(\mathbf{x})$ or $C \cup b_i \cup b_j \in Supp\, P(\mathbf{x})$. Since $|A \cap C| \geqslant 2$, we have $b_i \in C \cup b_i \setminus a' \in \mathcal{C}[A, B]$ or $b_i \in C \cup b_i \cup b_j \in \mathcal{C}[A, B]$. Therefore, by Lemma 20, (A, B) satisfies the 2-step axiom for b_i in $P(\mathbf{x})$.

Therefore, we showed that (A, B) satisfies the 2-step axiom in $P(\mathbf{x})$ for all $A, B \in Supp\, P(\mathbf{x})$ such that $|A| \leqslant |B|$.

Hence, it remains to show the 2-step axiom for (A, B) pairs such that $|A| > |B|$ and $|B| \in \{0, 1, 2\}$.

Subcase 5: $|A| > |B|$ and $|B| \leqslant 2$.

Subcase (5.1): $|B| = 1$ or $|B| = 0$.
Consider $P^*(\mathbf{x})$. Since $E \setminus A = B$ and $E \setminus B = A$, we have $B, A \in Supp\, P^*(\mathbf{x})$ and hence $deg\, P(\mathbf{x}) = deg\, P^*(\mathbf{x})$. Thus, by Subcase 1 and Subcase 2, (B, A) satisfies the 2-step axiom in $P^*(\mathbf{x})$. By Lemma 21, this is equivalent to saying that (A, B) satisfies the 2-step axiom in $P(\mathbf{x})$.

Subcase (5.2): $|A| \geqslant 4$ and $|B| = 2$.
When $\emptyset \notin Supp\, P(\mathbf{x})$ or $E \in Supp\, P(\mathbf{x})$, $deg\, P^*(\mathbf{x}) \leqslant deg\, P(\mathbf{x})$. Thus (A, B) has the 2-step axiom in $P(\mathbf{x})$ by Lemma 21. Now we assume $\emptyset \in Supp\, P(\mathbf{x})$ and $E \notin Supp\, P(\mathbf{x})$. By Lemma 20, there exists $C \in \mathcal{C}[A, B]$. As we showed in Subcase (3.2), we find $C \in \mathcal{C}[A, B]$ such that $|C \cap B|$ is maximum.

(1) $|C \cap B| = 1$.
By the same argument as in Subcase (3.2)(1), $|A \cap C| = 1$. By Lemma 20 (A, B) satisfies the 2-step axiom for any $a \in A \setminus C$ and any $b \in C \cap B$. Hence it suffices to show the 2-step axiom for $a' \in A \cap C$ and $b' \in B \setminus C$. But we can always find $C' \in \mathcal{C}[A, B]$ such that $a' \in A \setminus C'$ and $b' \in C' \cap B$.

(2) $|C \cap B| = 2$.
Just like Subcase (3.2)(2), there exists $C \in \mathcal{C}[A, B]$ such that $|C \cap A| = 2$. Since $C \cap B = B$, by Lemma 20, (A, B) satisfies the 2-step axiom for any b in B. Since $\emptyset \in Supp\, P(\mathbf{x})$, $(A, \emptyset)$

satisfies the 2-step axiom in $P^{\setminus B}(\mathbf{x})$ for any $a \in A$. This then implies that (A, B) satisfies the 2-step axiom for any $a \in A$ in $P(\mathbf{x})$.

Therefore, for any pair (A, B) of elements in *Supp* $P(\mathbf{x})$, the 2-step axiom holds, which contradicts our assumption that *Supp* $P(\mathbf{x})$ is not a jump system. Hence, by mathematical induction, *Supp* $P(\mathbf{x})$ is a jump system for any multilinear polynomial $P(\mathbf{x})$ in complex variables with complex coefficients which has definite parity and the HPP. $\square$

Now we generalize the hypothesis of $P(\mathbf{x})$ a little more by dropping the multilinearity condition. Then the technique of polarization which was introduced in Section 2 makes it possible to deduce a similar result on $P(\mathbf{x})$. Here is the other main theorem of this paper.

Theorem 22. *Let $P(\mathbf{x}) = \sum_{\alpha:\mathbf{Z}_n \to \mathbb{N}} a_\alpha \mathbf{x}^\alpha$ be a polynomial in n complex variables with definite parity, where $a_\alpha \in \mathbb{C}$. If $P(\mathbf{x})$ has the half-plane property, then Supp $P(\mathbf{x})$ is a jump system.*

Proof. Let (α, β) be a pair of arbitrarily chosen members of *Supp* $P(\mathbf{x})$. Then α and β can be expressed as n-tuples of nonnegative integers, $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $(\beta_1, \beta_2, \ldots, \beta_n)$, respectively. To prove the theorem, it suffices to show that (α, β) satisfies the 2-step axiom, i.e., for any step $u \in St(\alpha, \beta)$, from α to β, there exists $v \in St(\alpha + u, \beta)$ such that $\alpha + u + v \in$ *Supp* $P(\mathbf{x})$.

Consider $\mathcal{P}(P)$, the polarization of $P(\mathbf{x})$. By Proposition 9, $\mathcal{P}(P)$ also has the HPP and since $\mathcal{P}(P)$ is a multiaffine polynomial with definite parity, by Theorem 2, *Supp* $\mathcal{P}(P)$ is a jump system. Let the ground set E of $P(\mathbf{x})$ be $\mathbb{Z}_n := \{1, 2, \ldots, n\}$. Then the ground set of $\mathcal{P}(P)$ is $\bigcup_{i=1}^n \{(i)_j : j = 1, 2, \ldots, r_i\}$ where $r_i = deg_i\, P(\mathbf{x})$. Hence, given α and β in *Supp* $P(\mathbf{x})$, it is clear that $\bigcup_{i=1}^n \{(i)_j : j = 1, 2, \ldots, \alpha_i\}(:= A)$ and $\bigcup_{i=1}^n \{(i)_j : j = 1, 2, \ldots, \beta_i\}(:= B)$ are contained in *Supp* $\mathcal{P}(P)$.

Now using the fact that (A, B) satisfies the 2-step axiom, we will show that (α, β) also satisfies the 2-step axiom.

Let $u \in St(\alpha, \beta)$. Then u is either of the form $-\delta_k$ for some $k \in E$ such that c or of the form δ_h for some $h \in E$ such that $\alpha_h < \beta_h$. Here, δ_l is the n-tuple whose only nonzero entry is the lth entry and it is 1.

(1) When $u = -\delta_k$ for some $k \in E$ such that $\alpha_k > \beta_k$, we consider an element $a := (k)_{\beta_k+1} \in A$. Since (A, B) satisfies the 2-step axiom and $a \in A \setminus B$, there exists $a' \in A \setminus B \setminus a$ such that $A \setminus a \setminus a' \in$ *Supp* $\mathcal{P}(P)$ or there exists $b' \in B \setminus A$ such that $A \setminus a \cup b' \in$ *Supp* $\mathcal{P}(P)$.

(i) If there exists $a' \in A \setminus B \setminus a$, then either we have $a' \in \{(k)_{\beta_k+2}, \ldots, (k)_{\alpha_k}\}$ or we have $a' \in \{(i)_{\beta_i+1}, \ldots, (i)_{\alpha_i}\}$ for some $i \neq k$ such that $\alpha_i > \beta_i$. Therefore, the former case implies that $v = -\delta_k \in St(\alpha + u, \beta)$ and *Supp* $P(\mathbf{x})$ contains $\alpha - \delta_k - \delta_k$. The latter case implies $v := -\delta_i \in St(\alpha + u, \beta)$ and $\alpha - \delta_k - \delta_i \in$ *Supp* $P(\mathbf{x})$.

(ii) If there exists $b' \in B \setminus A$, then for some i such that $\alpha_i < \beta_i$ we have $b' \in \{(i)_{\alpha_i+1}, \ldots, (i)_{\beta_i}\}$. Thus, $A \setminus a \cup b' \in$ *Supp* $\mathcal{P}(P)$ implies that $v := \delta_i \in St(\alpha + u, \beta)$ and $\alpha - \delta_k + \delta_i \in$ *Supp* $P(\mathbf{x})$.

(2) When $u = +\delta_h$ for some $h \in E$ such that $\alpha_h < \beta_h$, we consider an element $b := (h)_{\alpha_h+1} \in B$. Since $b \in B \setminus A$, there exists $a'' \in A \setminus B$ such that $A \cup b \setminus a'' \in$ *Supp* $\mathcal{P}(P)$ or there exists $b'' \in B \setminus A \setminus b$ such that $A \cup b \cup b'' \in$ *Supp* $\mathcal{P}(P)$.

(i) If there exists $a'' \in A \setminus B$, we have $a'' \in \{(i)_{\beta_i+1}, \ldots, (i)_{\alpha_i}\}$ for some i such that $\alpha_i > \beta_i$. Hence $v := -\delta_i \in St(\alpha+u, \beta)$. $A \cup b \setminus a'' \in Supp\,\mathcal{P}(P)$ implies that $\alpha + \delta_h - \delta_i \in Supp\,P(\mathbf{x})$.

(ii) If instead there exists $b'' \in A \setminus B \setminus b$, then there exists either $b'' \in \{(h)_{\alpha_h+2}, \ldots, (h)_{\beta_h}\}$ or $b'' \in \{(i)_{\alpha_i+1}, \ldots, (i)_{\beta_i}\}$ for some i such that $\alpha_i < \beta_i$. Hence this implies that δ_h or δ_i is in $St(\alpha + u, \beta)$. Since $A \cup b \cup b'' \in Supp\,\mathcal{P}(P)$, we have either $\alpha + \delta_h + \delta_h \in Supp\,P(\mathbf{x})$ or $\alpha + \delta_h + \delta_i \in Supp\,P(\mathbf{x})$.

Hence, for any $u \in St(\alpha, \beta)$, there exists $v \in St(\alpha+u, \beta)$ such that $\alpha+u+v \in Supp\,P(\mathbf{x})$, that is, (α, β) satisfies the 2-step axiom. Therefore $Supp\,P(\mathbf{x})$ is a jump system. $\square$

Let $P(\mathbf{x})$ be a homogeneous polynomial of degree r, then obviously P has definite parity. Therefore, we have the following corollary.

Corollary 23 (*Choe et al.* [3] *and Choe* [4]). *Let* $P(\mathbf{x}) = \sum_{\alpha \in \mathcal{S}_r} a_\alpha \mathbf{x}^\alpha$ *be a homogeneous degree-r polynomial with the half-plane property with complex coefficients. Then Supp* $P(\mathbf{x})$ *is a jump system.*

Moreover if we assume that $P(\mathbf{x})$ is multiaffine as well as homogeneous, all the elements in the support are subsets of E with the same size. Then the 2-step axiom is equivalent to the bases exchange axiom of in matroids. Therefore, *Supp* $P(\mathbf{x})$ constitute a special kind of jump system, which is a matroid.

Corollary 24 (*Choe et al.* [3] *and Choe* [4]). *If a homogeneous and multilinear polynomial* $P(\mathbf{x}) = \sum_{S \in \mathcal{B}} a_S \mathbf{x}^S$ *of degree r with* $a_S \in \mathbb{C}$ *has the half-plane property, then there exists a matroid M such that the set of its bases is exactly* $\mathcal{B}$.

4. Open problems

The problems related to the HPP have a close connection with reliability theory [7,8]. Given a connected undirected graph $G := (V, E)$, let p_e be the probability that the edge $e \in E$ is operating. We assume each edge is independently operational. Then the reliability $Rel_G(p)$ of G is the probability that the operating part of G is connected. Hence, $Rel_G(\mathbf{x})$ is a polynomial in $\mathbf{x}$ and can be interpreted as a generating polynomial given below:

$$\begin{aligned} Rel_G(\mathbf{x}) &= \sum_{A \subseteq E,\ G(A)\ connected} \mathbf{x}^A (1-\mathbf{x})^{E \setminus A} \\ &= (1-\mathbf{x})^E \sum_{A \subseteq E,\ G(A)\ connected} \prod_{e \in A} \frac{x_e}{1-x_e}, \end{aligned}$$

where $G(A)$ denotes the subgraph of G induced by A.

Conversely, the relation between the generating polynomial for the set of all connected subgraphs and the reliability polynomial can be expressed as

$$C_G(\mathbf{x}) := \sum_{A\subset E,\ G(A) connected} \mathbf{x}^A$$
$$= \prod_{e\in E} (1+x_e) Rel_G\left(\frac{x}{x+1}\right).$$

Brown and Colbourn [2] conjectured that the roots of the reliability polynomial for any connected graph are located in the closed disc $|\mathbf{x}-1| < 1$.

Conjecture 25 (*Multivariate Brown–Colbourn conjecture, Sokal [14]*). *Let G be a connected loopless graph. If* $\operatorname{Re} x_e > -\frac{1}{2}$ *for all* $e \in E$, *then* $P_{\mathcal{I}^*}(\mathbf{x}) \neq 0$, *where* $\mathcal{I}^*$ *is the set of all independent sets of the cographic matroid* $M^*(G)$.

For a polynomial P, we say that P has the Brown–Colbourn property if $P \in \mathcal{F}_{H^n_{\pi,1/2}}$, where $H_{\theta,K} := \{x \in \mathbb{C} : \operatorname{Re}(e^{-i\theta x}) > K\}$ and $\mathcal{F}_D$ denotes the set of all functions f that are analytic in D and are either nonzero in D or $f \equiv 0$. The following proposition indicates the close relation between polynomials with the HPP and the Brown–Colbourn property.

Proposition 26 (*Choe et al. [3, Corollary 2.3]*). *Let M be a matroid. If the independent-set generating polynomial* $P_{\mathcal{I}}(\mathbf{x})$ *has the Brown–Colbourn property, then the basis generating polynomial* $P_{\mathcal{B}}$ *has the half-plane property*.

Conjecture 25 has been proven for series-parallel graphs [14,16]. But recently, Royle and Sokal found counterexamples to the Brown–Colbourn conjecture [13].

Conjecture 27 (*Last stand Brown–Colbourn conjecture*). *If G is a three-connected simple graph and* $q_e = q$ *for all* $e \in E$, *then all roots of* $Rel_G(q)$ *satisfy* $|q| \leqslant 1$.

We may further attempt to generalize the support theorem as follows:

Question 28 (*Choe et al. [3]*). *If a multiaffine polynomial P with n complex variables has the half-plane property, is the support of P a delta-matroid*? *Can we prove the same-phase property for such a polynomial*?

One problem here is that the same phase property does not hold in general. Note that $1+\lambda x_i$ has the HPP for any $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$. So $\prod_{i=1}^n (1+\lambda_i x_i)$ has the HPP for any $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ with all $Re\ \lambda_i > 0$. This need not have the same phase property.

Question 29 (*D.G.Wagner*). *Let* $P(x_1, \ldots, x_n) = \sum_\alpha a_\alpha \mathbf{x}^\alpha$ *have the half-plane property. Then at least one of the following two conditions must hold.*

- *P has the same-phase property*.
- *P has a linear factor* $1+\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$ *with* $\operatorname{Re}\lambda_i \geqslant 0$ *for all* $1 \leqslant i \leqslant n$.

A solution to this would give a best possible answer to same-phase properties in general.

As we can see from the counter examples such as the Fano matroid, the non-Fano matroid, and Pappus and non-Pappus matroids, not all matroids have the HPP. But there are quite a few classes of matroids that have the HPP. Those matroids are $\sqrt[6]{1}$-root-of-unity matroids (therefore, regular, graphic and cographic matroids), uniform matroids, some transversal matroids, and all matroids of rank or corank at most 2 and so on [3].

Question 30. *Are there other classes of matroids with the half-plane property*? *Are there any other matroids which do not have the half-plane property*?

Given a set system $\mathcal{S}$, we say that $\mathcal{S}$ has *the weak HPP* if there exists a polynomial P with the HPP whose support is $\mathcal{S}$. It is known that every matroid that is representable over $\mathbb{C}$ has the weak HPP [3].

Question 31 (*Choe et al. [3]*). *Does every matroid M have the weak half-plane property*? *And if not, which ones do*?

We have seen that the collection of homogeneous multiaffine polynomial with the HPP is closed under deletion, contraction and taking the dual. Furthermore, the collection of all matroids with the HPP is closed under minors, duality, direct sums, 2-sums, series connection, and parallel connection, principal truncation and principal cotruncation, and etc, [3].

Question 32. *Are there any other operations on matroids which preserve the half-plane property*?

It is not completely know yet for rank 3 matroids with 7 elements whether they have the HPP or not.

Question 33. *What are the minor-minimal binary nonhalf-plane property matroids*? *Are there any other such matroids than F_7 and F_7^**?

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